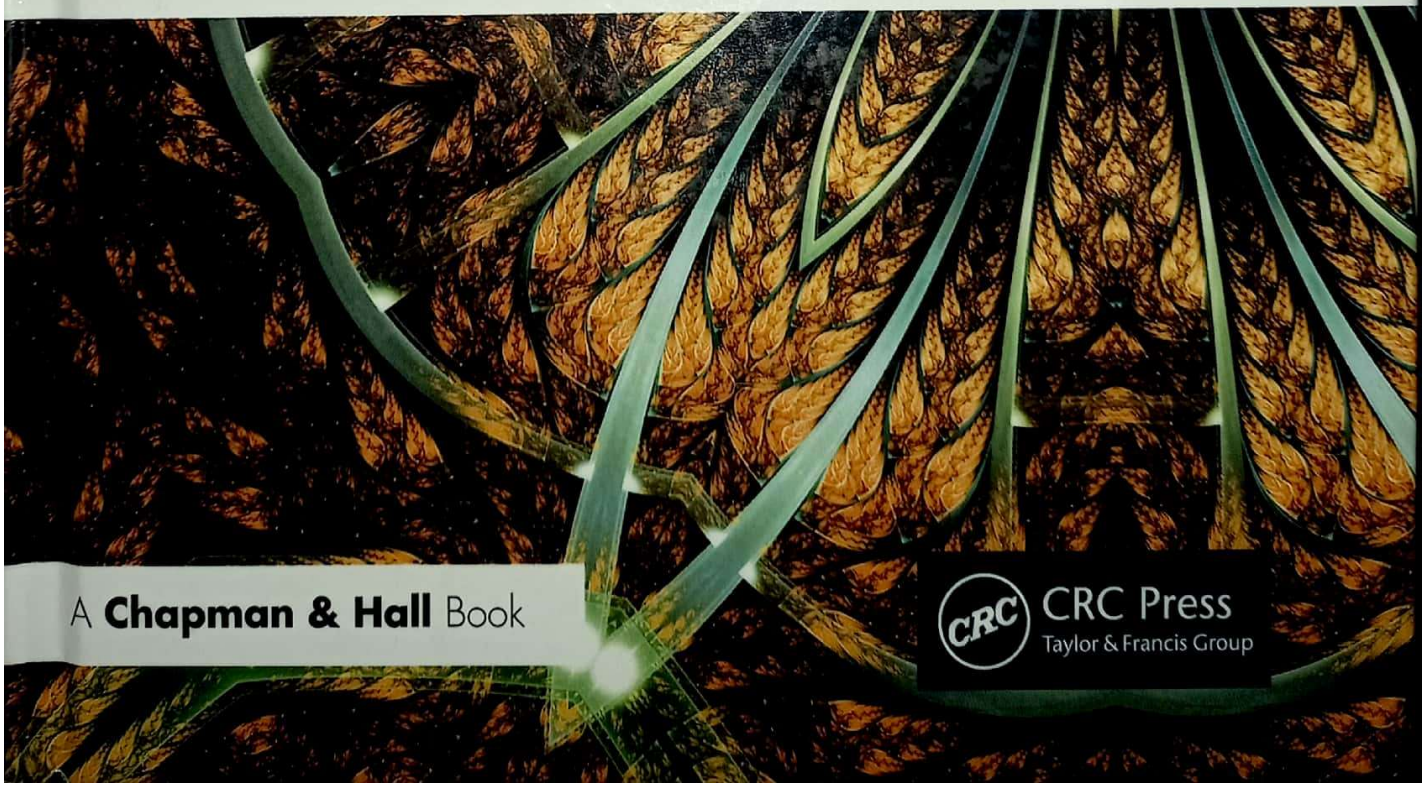
AN INTRODUCTION TO METRIC SPACES

Dhananjay Gopal, Aniruddha Deshmukh,
Abhay S. Ranadive and Shubham Yadav



A **Chapman & Hall** Book

CRC Press
Taylor & Francis Group

Scanned by CamScanner

First edition published 2021
by CRC Press
6000 Broken Sound Parkway NW, Suite 300, Boca Raton, FL 33487-2742

and by CRC Press
2 Park Square, Milton Park, Abingdon, Oxon, OX14 4RN

© 2020 Taylor & Francis Group, LLC

CRC Press is an imprint of Taylor & Francis Group, LLC

Reasonable efforts have been made to publish reliable data and information, but the author and publisher cannot assume responsibility for the validity of all materials or the consequences of their use. The authors and publishers have attempted to trace the copyright holders of all material reproduced in this publication and apologize to copyright holders if permission to publish in this form has not been obtained. If any copyright material has not been acknowledged please write and let us know so we may rectify in any future reprint.

Except as permitted under U.S. Copyright Law, no part of this book may be reprinted, reproduced, transmitted, or utilized in any form by any electronic, mechanical, or other means, now known or hereafter invented, including photocopying, microfilming, and recording, or in any information storage or retrieval system, without written permission from the publishers.

For permission to photocopy or use material electronically from this work, access www.copyright.com or contact the Copyright Clearance Center, Inc. (CCC), 222 Rosewood Drive, Danvers, MA 01923, 978-750-8400. For works that are not available on CCC please contact mpkbookspermissions@tandf.co.uk

Trademark notice: Product or corporate names may be trademarks or registered trademarks, and are used only for identification and explanation without intent to infringe.

ISBN: 978-0-367-49348-6 (hbk)
ISBN: 978-1-003-04587-8 (ebk)

Typeset in CMR10
by Lumina Datamatics Limited

https://www.routledge.com/An-Introduction-to-Metric-Spaces/Gopal-Deshmukh-Ranadive-Yadav/p/book/9780367493486

Printed in the United Kingdom

Scanned by CamScanner

Contents

Preface ix

A Note to the Reader xiii

Authors xv

1 Set Theory 1
- 1.1 Sets . . . 1
  - 1.1.1 The empty set . . . 4
  - 1.1.2 Operations on sets . . . 5
  - 1.1.3 Uniqueness of the empty set . . . 9
  - 1.1.4 Power sets . . . 9
  - 1.1.5 Cartesian products . . . 9
- 1.2 Relations . . . 10
  - 1.2.1 Types of relations . . . 12
  - 1.2.2 Equivalence relations . . . 13
  - 1.2.3 Partition of sets . . . 15
  - 1.2.4 Partial order relations . . . 16
- 1.3 Functions . . . 22
  - 1.3.1 Composition of functions . . . 24
  - 1.3.2 Inverse of a function . . . 26
  - 1.3.3 Images of sets under functions . . . 32
  - 1.3.4 Inverse images of sets under functions . . . 36
- 1.4 Countability of Sets . . . 39
  - 1.4.1 Finite sets . . . 41
  - 1.4.2 Countable sets . . . 44

Problem Set . . . 47

Biographical Notes . . . 51

2 Metric Spaces 55
- 2.1 Review of Real Number System and Absolute Value . . . 55
- 2.2 Young, Hölder, and Minkowski Inequalities . . . 57
- 2.3 Notion of Metric Space . . . 64
- 2.4 Open Sets . . . 81
  - 2.4.1 Subspace topology . . . 96
  - 2.4.2 Product topology . . . 97

Scanned by CamScanner

2.5 Closed Sets
2.6 Interior, Exterior, and Boundary Points
2.7 Limit and Cluster Points
2.8 Bounded Sets
2.9 Distance Between Sets
2.10 Equivalent Metrics
Problem Set
Biographical Notes

3 Complete Metric Spaces
3.1 Sequences
3.1.1 Subsequences
3.2 Convergence of Sequence
3.3 Complete Metric Spaces
3.4 Completion of Metric Spaces
3.4.1 Construction of the set Z
3.4.2 Embedding X in Z
3.4.3 Proving Z is complete
3.4.4 Uniqueness of extension up to isometry
3.5 Baire Category Theorem
3.5.1 Category of sets
3.5.2 Baire category theorem
3.5.3 Applications of Baire category theorem
Problem Set
Biographical Notes

4 Compact Metric Spaces
4.1 Open Cover and Compact Sets 161
4.2 General Properties of Compact Sets 161
4.3 Sufficient Conditions for Compactness 165
4.4 Sequential Compactness 169
4.5 Compactness: Characterizations 172
Problem Set 174
Biographical Notes 179

5 Connected Spaces 180
5.1 Connectedness 183
5.1.1 Connected subsets 183
5.2 Components 185
5.3 Totally Disconnected Spaces 190
Problem Set 192
193

Scanned by CamScanner

6 Continuity **195**
6.1 Continuity of Real Valued Functions 195
6.2 Continuous Functions in Arbitrary Metric Spaces 197
6.2.1 Equivalent definitions of continuity and other characterizations 202
6.2.2 Results on continuity 210
6.3 Uniform Continuity 217
6.4 Continuous Functions on Compact Spaces 224
6.5 Continuous Functions on Connected Spaces 229
6.5.1 Path connectedness 237
6.6 Equicontinuity and Arzela-Ascoli's Theorem 242
6.7 Open and Closed Maps 245
6.8 Homeomorphism 246
Problem Set 251
Biographical Notes 252

7 Banach Fixed Point Theorem and Its Applications **255**
7.1 Banach Contraction Theorem 255
7.2 Applications of Banach Contraction Principle 260
7.2.1 Root finding problem 260
7.2.2 Solution of system of linear algebraic equations 261
7.2.3 Picard existence theorem for differential equations . . 264
7.2.4 Solutions of integral equations 267
7.2.5 Solutions of initial value and boundary value problems 271
7.2.6 Implicit function theorem 273
Problem Set 274
Biographical Notes 276

Appendix A **277**

Bibliography **281**

Index **283**

Scanned by CamScanner