To find the sum of the forces on the element, the ⊥ pressure on each face must be evaluated.

$P_1 = P|_x, \quad P_2 = P|_{x+\Delta x}$ . . . . So on

force due to gravity $\rho g(\Delta x . \Delta y . \Delta z) . g$ –

$\sum F = 0$ (fluid at rest)

$$\rho g(\Delta x . \Delta y . \Delta z) + (P|_x - P|_{x+\Delta x}) . \underline{e_x} . \Delta y \Delta z + (P|_y - P|_{y+\Delta y}) . \underline{e_y} . \Delta x . \Delta z + (P|_z - P|_{z+\Delta z}) \underline{e_z} . \Delta x . \Delta y = 0$$

lim $\Delta x, \Delta y, \Delta z \to 0$

$$\rho g - \frac{P|_{x+\Delta x} - P|_x}{\Delta x} \underline{e_x} - \frac{P|_{y+\Delta y} - P|_y}{\Delta y} \underline{e_y} - \frac{P|_{z+\Delta z} - P|_z}{\Delta z} \underline{e_z}$$

$$\rho \underline{g} - \frac{\partial P}{\partial x} . e_x - \frac{\partial P}{\partial y} . e_y - \frac{\partial P}{\partial z} . e_z = 0$$

or $\rho g - (\nabla P) = 0$

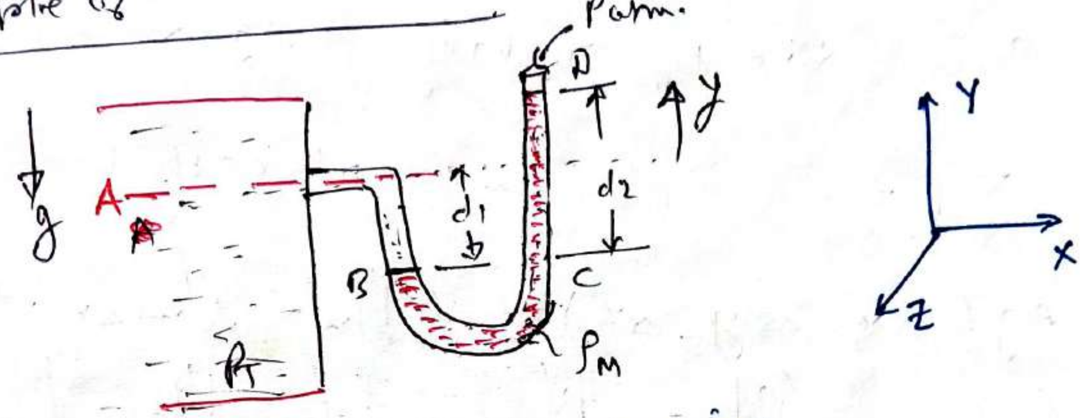
or $\boxed{\rho \bar{g} = \bar{\nabla} P}$ — (A) $\nabla \to$ del operator. (bar to denote vector)

Basic eqn of fluid in statics and this says that the maximum rate of change in the pressure occurs in direction gravitation vector

* The isolines are ⊥ to the gradient. ~~constant~~
* ~~pressure lines are~~ constant pressure lines are ⊥ to gravitational vector.
* point to point variation may be obtained by integrating the eqn $\underline{A}$ –

point to point variation in the pressure can be obtained by ~~solving~~ integrating the above eqn

Example of Manometer



only in y direction pressure is changing

$\therefore \frac{dP}{dy} e_y$ ~~$\frac{\partial P}{\partial y} e_y$~~ $= -\rho g e_y$

integrating b/w C & D

$P_{atm} - P_C = -\rho_m \cdot g \cdot d_2$

integrating ~~b/w~~ b/w A & B

$P_A - P_B = -\rho_T g \cdot d_1$

~~$P_A - P_{atm} = P_m$~~

B & C are at the same level.

$P_B = P_C$

$\therefore P_A - P_{atm} = \rho_m g d_2 - \rho_T \cdot g \cdot d_1$

gage pressure

Conclusion: Remark: behaviour of static fluid has been examined. Application of Newton's law led to the ~~description~~

description

* of the point to point variation in fluid pressure, from which force relation were developed.

Momentum Transfer

Fluid Motion

weltty

Basic laws of fluid motion.

Law	Equation
1. The law of mass conservation	Continuity eqn
2. Newton's II law of motion	Momentum theorem.
3. The first law of Thermodynamics	Energy eqn.

geankp

General Molecular transport Equation for Momentum, Heat & Mass Transfer.

U3- and General principal of property balance

Transport Process: we are concerned of transfer of a property (mass, momentum, energy) by a

molecular transport

- Each molecules have certain amount of heat, mass & momentum associated with it.
- descrepency in this properties within molecule leads to transport of these properties
- more dense less transport (liquids)
- less dense more transport (gases)

General molecular transport Equation: All these three molecular transport process are characterized in elementary sense by the same general type of equation.

$$\text{Rate of a transfer process} = \frac{\text{driving force}}{\text{resistance}}$$

remember $\boxed{I = \frac{V}{R}}$ ohm's law.

- Need a driving force to overcome resistance.

in general.

$$\psi_z = -\delta \frac{d\theta}{dz}$$

$\psi_z \rightarrow$ flux of the molecular property being transferred.

$$\frac{\text{flow}}{\text{Area} \cdot \text{Time}}$$

$z \rightarrow$ direction of transport

$\delta \rightarrow$ Proportionality constant

$\theta \rightarrow$ concentration of the property (mass, heat, momentum)
m T u

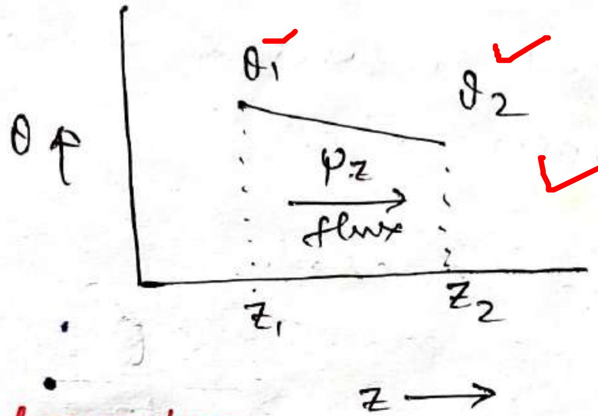
for steady state (s.s.)

$\psi_z =$ constant

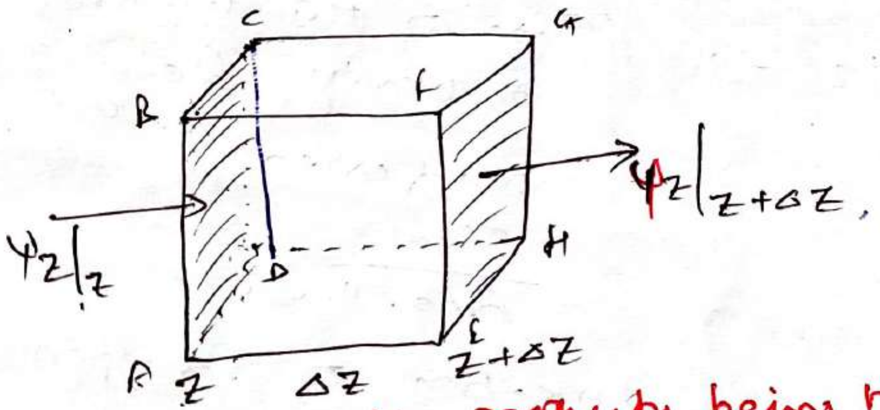
q = K λ

$$\psi_z \int_{z_1}^{z_2} dz = -\delta \int_{\theta_1}^{\theta_2} d\theta$$

$$\psi_z = -\delta \frac{(\theta_2 - \theta_1)}{z_2 - z_1}$$



General property balance for Unsteady state



To account for the property being transported in the * entire system, a general property balance or conservation eqn for the property at unsteady state is read.

(rate in of ψ) + (rate of generation of ψ) = (rate out of ψ) + Accumulation of ψ

ψ → T, v.

Assume the x-n of area of the element is 1

∴ volume = (Δ . 1)

plus / Arrgument

$$1 \cdot \psi|_z + R^{(\Delta z \cdot 1)} = \psi|_{z+\Delta z} \cdot 1 + \frac{\partial \theta}{\partial t} \cdot (\Delta z \cdot 1)$$

$\theta \rightarrow$ conc / m^3 vol.

$R \rightarrow$ generation / m^3 vol.

Divide by Δz

$$\frac{\partial \theta}{\partial t} = R - \left(\frac{\psi|_{z+\Delta z} - \psi|_z}{\Delta z} \right)$$

$$\boxed{\frac{\partial \theta}{\partial t} = R - \frac{\partial \psi}{\partial z}}$$

$\Delta z \rightarrow 0$

Now $\psi = -\delta \frac{\partial \theta}{\partial z}$

$$\frac{\partial \theta}{\partial t} = R + \delta \frac{\partial^2 \theta}{\partial z^2}$$ (Assume $\delta \rightarrow$ constant)

$$\boxed{\frac{\partial \theta}{\partial t} - \delta \frac{\partial^2 \theta}{\partial z^2} = R}$$

If No generation then

$$\boxed{\frac{\partial \theta}{\partial t} = \delta \frac{\partial^2 \theta}{\partial z^2}}$$

Take example 2.3-1 from geanko

These are general equation for the conservation of momentum, thermal energy or mass and ~~will be used~~. It is applicable only for molecular transport and donot consider other transport Mechanism.

Similarity in Momentum, heat & Mass transfer | Skip (18)

geankoplis

Momentum transport & Newton's law:

Consider a fluid has x directed momentum

$v_x \cdot \rho \longrightarrow \frac{\text{momentum}}{m^3}$ in $x \rightarrow$

This momentum due to difference in momentum of different layers, is being transferred in z direction. Why?? As this transfer actually due to random motion of molecules in z direction (+ive & -ive) b/w faster moving and slower moving layer.

$$\tau_{zx} = -\nu \frac{d(v_x \cdot \rho)}{dz}$$

This eqn describes the transfer of momentum.

$\nu \rightarrow$ kinematic viscosity | momentum diffusivity m^2/s

$\tau_{zx} \rightarrow$ flux of x directed momentum the z direction

$\nu \rightarrow$ momentum diffusivity $= (\mu/\rho)$

Heat Transfer & Fourier's law: Fourier's law for molecular transport of heat ~~or heat~~ conduction in a fluid ~~or~~ solid ~~for heat conduction in solid.~~ is

$$\frac{q_z}{A} = -k \frac{dT}{dz}$$

if we have constant ρc_p

Then

$$\frac{q_z}{A} = -\left(\frac{k}{\rho c_p}\right)\left(\frac{d(\rho c_p T)}{dz}\right)$$

$\longrightarrow \alpha \rightarrow$ thermal diffusivity

$\rho c_p T \rightarrow$ concentration of heat or thermal energy

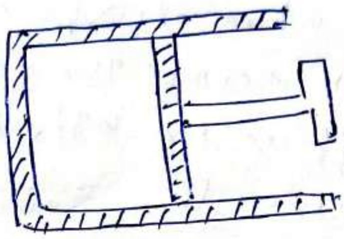
Transfer is due to the molecular diffusion

when there is temperature ~~difference~~ gradient in a fluid equal numbers of molecules diffuse in each direction b/w hot & cold region

Overall Mass Balance and Continuity equation

* As a first step in the solution of flow problem principle of mass conservation is applied.
* Conservation laws are defined for the system. System: Means the collection of matter of fixed identity.
* In flow problem's identity of particles are not fixed therefore system can not be defined as such control volume concept is used.

Ex. of System — piston cylinder from thermodynamics



control volume: is a region fixed in space through which fluid flows.

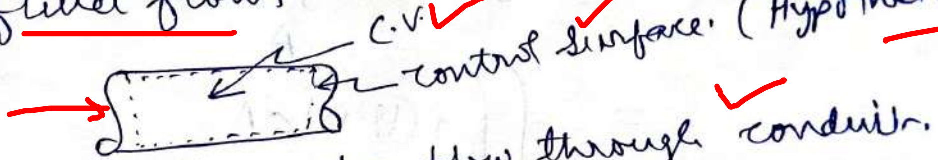


Fig: control volume for flow through conduit.

Overall mass balance equation for fluid flow:

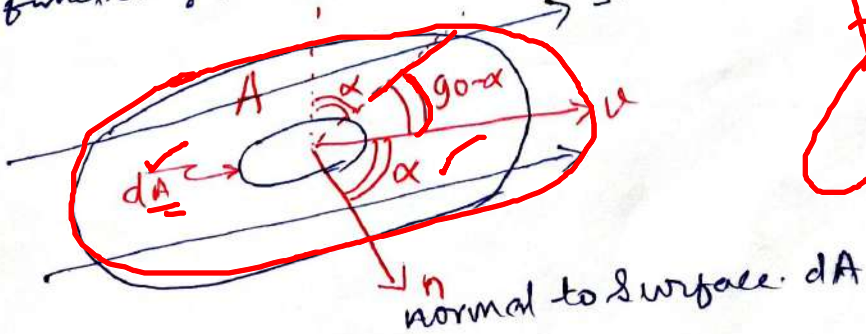
Overall mass balance equation for a general control volume, where no mass is being generated is as follows

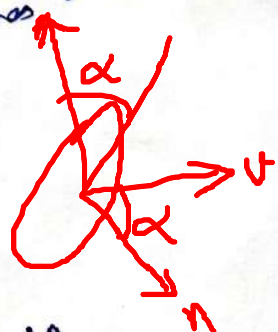
U.S.S

(Rate of mass out from C.V.) − (Rate of mass in to C.V.) + Rate of mass acc. in C.V. = 0

consider a general control volume located in a flow field (Field refers to a quantity defined as a function of position and time throughout a given region)

here you may explain Eulerian & Lagrangian approach of defining field.





consider the differential area dA on the control surface.

rate of mass efflux from this elemental area

$$= (\rho \cdot v \, dA \cos\alpha)$$

dA cos α is the projection of dA over vertical plane (normal to velocity vector)

$$\rho(v \, dA \cos\alpha) = \rho(v \cdot n) \, dA$$

Therefore for the entire surface A net rate of mass out flow across the control surface or net mass efflux in (kg/s) from the entire control volume V

$$\text{net mass efflux from control volume} = \iint_A v\rho \cos\alpha \, dA$$

$$= \iint_A \rho(\vec{v} \cdot \vec{n}) \, dA$$

Note - if mass is entering the c.v. $\alpha > 90^\circ$

hence $\cos\alpha = \ominus$ive (i.e mass influx) [not written above "mass"]

if mass is flowing out $\alpha < 90^\circ$ (not mass efflux)

Rate of accumulation of mass within c.v. "V"

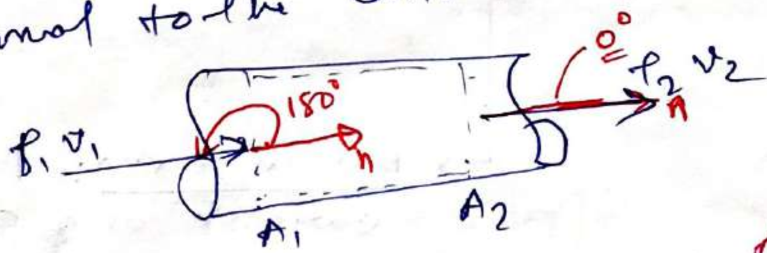
$$= \frac{\partial}{\partial t} \iiint_V \rho \, dV = \frac{dM}{dt}$$

M → mass of the fluid in the volume in Euler

General form of the eqn

$$\iint_A \rho (\bar{v} \cdot \hat{n}) \, dA + \frac{\partial}{\partial t} \iiint_V \rho \, dV = 0 \quad \text{(no generation)}$$

★ Consider a case where the flow is steady and normal to the control surface A_1 & A_2



Why ?? cos will take compo...

$$\therefore \iint_A v\rho \cos\alpha \, dA = \underbrace{\iint_{A_2} v\rho \cos\alpha_2 \, dA}_{\text{Out flow}} + \underbrace{\iint_{A_1} v\rho \cos\alpha_1 \, dA}_{\text{inflow}}$$

at A_1, $\alpha_1 = 180^\circ$ & at A_2 $\alpha_2 = 0^\circ$

$$\iint_A v\rho \cos\alpha \, dA = v_2 \rho_2 A_2 - v_1 \rho_1 A_1 = 0$$

$$\therefore \boxed{v_2 \rho_2 A_2 = v_1 \rho_1 A_1}$$

P.T.O.

206

For a component overall balance eqn, general mass balance eqn can be extended as

$$m_{i_2} - m_{i_1} + \frac{dM_i}{dt} = R_i$$

m_i → component flow rate

M_i → kg of component in the C.V.

Eularian & Lagrangian approaches: (Welty 22/29)

Eularian and Lagrangian: The term field refer to a quantity defined as a function of position and time through a given region.

There are two different forms of representing field in fluid mechanics →

→ ~~Lagrangian~~ Lagranges form

Particle identity is fixed and particle coordinates are functions of time

→ Euler form: The value of a fluid variable at a given point and at a given time. In functional form.

$$v = v(x, y, z, t)$$

Where x,y,z and t are all independent variable. For a particular (x_1, y_1,z_1) and t_1 the above equation gives the value of velocity of the field at that position and at time t_1.This is most common form of presenting the velcoity field.

Consider steady state flow $\frac{dM}{dt} \to 0$

$$\iint_A \rho v \cos\alpha \, dA = \iint_{A_2} v\rho \cos\alpha_2 \, dA + \iint_{A_1} \rho v \cos\alpha_1 \, dA$$

$$= v_2 \rho_2 A_2 - \rho_1 v_1 A_1 = 0$$

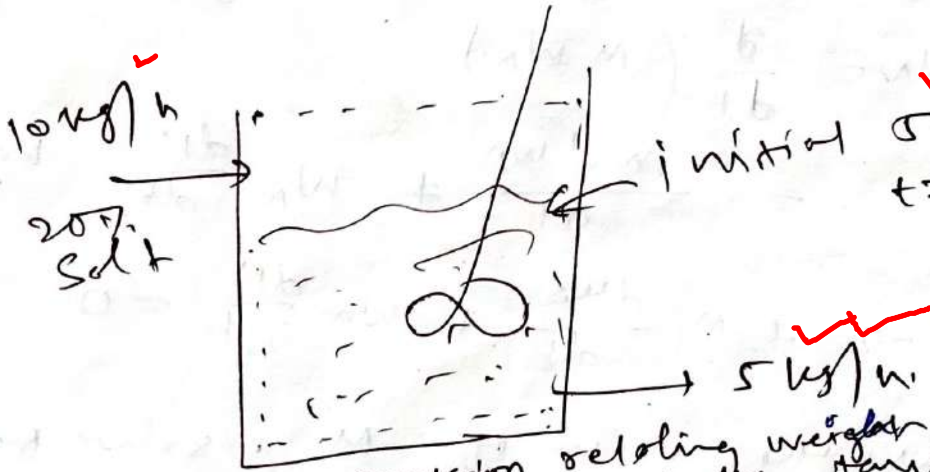
$\because$ at A_1 $\alpha_1 = 180°$

at A_2 $\alpha_2 = 0°$

or

$$\boxed{v_2 \rho_2 A_2 = \rho_1 v_1 A_1}$$

Overall Mass balance in a stirred Tank



Q Derive an expression relating weight fraction w_A of the salt in the tank at any time t after

$$\iint_A v\rho \cos\alpha \, dA = m_2 - m_1 = 5 - 10 = -5 \text{ kg sol}/h$$

$$\frac{dM}{dt} = \frac{\partial}{\partial t} \iiint_V \rho \, dv$$

Balance eq^n

$$-5 + \frac{dM}{dt} = 0$$

$$\frac{dM}{dt} = 5$$

$$\int_{500}^{M} = 5\int_0^t$$

$$\underline{\underline{M = 5t + 500}}$$ overall balance

<u>Component balance</u>

$$\iint_A v\rho \cos\alpha \, dA = 5(w_A) - 10(0.2)$$

$$= 5w_A - 2 \quad \text{kg/hr} \quad \text{—(1)}$$

$$\frac{\partial}{\partial t}\iiint_V \rho \, dV = \frac{d}{dt}(M w_A)$$

$$= \frac{M \, dw_A}{dt} + w_A \frac{dM}{dt} \quad \frac{\text{kg Salt}}{\text{h.}} \quad \text{—(2)}$$

(1) + (2)

$$5w_A - 2 + M\frac{dw_A}{dt} + w_A\frac{dM}{dt} = 0$$

put the value of M & solve for $\underline{\underline{w_A}}$

$\underline{\underline{B.C}}$ $\quad w_A = 0.1$ at $t = 0$,
$\quad w_A = w_A$ at $t = t$

$$5w_A - 2 + (500 + 5t)\frac{dw_A}{dt} + 5w_A = 0$$

$$\int_{w_A=0.1}^{w_A} \frac{dw_A}{2-10w_A} = \int_{t=0}^{t} \frac{dt}{500+5t}$$

$$-\frac{1}{10} \ln\left(\frac{2-10w_A}{1}\right) = \frac{1}{5}\ln\left(\frac{500+5t}{500}\right)$$

or

$$w_A = -0.1\left(\frac{100}{100+t}\right)^2 + 0.20$$

Avg velocity to use in overall mass balance

$$V_{av} = \frac{1}{A}\iint_A v\, dA$$ | if v varies over the x-sect area

avg or bulk velocity for a surface over which v is normal to A and density ρ is assumed constant.

Ex. Variation of velocity across control surface and Avg velocity.

Consider incompressible fluid (ρ is constant) flow through a circular pipe of radius R

$$v = v_{max}\left(1-\left(\frac{r}{R}\right)^2\right)$$ (Parabolic profile for laminar flow)

$$v_{av} = \frac{\int_0^{2\pi}\int_0^{R} v_{max}\left(1-\left(\frac{r}{R}\right)^2\right) r\,dr\,d\theta}{\pi R^2}$$

$$v_{avg} = \frac{v_{max}}{\pi R^4} \int_0^{2\pi} \int_0^R (R^2 - r^2)\, r\, dr\, d\theta$$

$$v_{av} = \frac{v_{max}}{\pi R^2} (2\pi - 0) \left(\frac{R^4}{2} - \frac{R^4}{4} \right)$$

$$\boxed{v_{av} = \frac{v_{max}}{2}}$$

Overall Energy Balance

law of conservation of energy combined with the first law of thermodynamics

First of law of thermo.

$$\boxed{\Delta E = Q - W}$$

E → Total Energy / mass of fluid

Q → heat absorbed / mass of fluid

W → work of all kinds done / mass of fluid on surrounding

Overall Energy balance eqn

rate of entity out − rate of entity in + rate of entity accumulation = 0

entity = energy

E = f(potential, kinetic, internal - - -)

potential → position

kinetic → translational or rotational motion of mass

internal → all other energy as ~~bonding~~ ~~vibrational~~ ~~etc.~~ bonding energy)

E → Energy present in a system (J/kg)

Energy per unit mass ~~Per unit mass~~

$$E = U + \frac{v^2}{2} + zg$$

Accumulation $= \frac{\partial}{\partial t} \iiint \left(U + \frac{v^2}{2} + zg\right) \rho \, dV$

Apart from U, $\frac{v^2}{2}$, & zg energy is transferred as the mass flows into and out of the C.V.. This pressure-volume [illegible] per unit mass of fluid is PV

Combining the 'U' & 'PV' terms

$\therefore H = U + PV$ → Enthalpy

$\therefore$ Total energy carried with a unit mass $= H + \frac{v^2}{2} + zg$

Net energy efflux from control volume

$$= \iint_A \left(H + \frac{v^2}{2} + zg\right) \cdot \rho v \cos\alpha \, dA$$

dA & α are as earlier.

This accounts for all energy associated with mass in the system and moving across the boundary in the entity balance.

Now consider heat and work energy.

(a) q → heat transferred per unit ~~time~~ to the fluid across the boundary because of temperature gradient. q → ⊕ ive

(b) W → shaft work + pressure volume work
W_s (Purely mechanical) — already used in H

As per convention
Note, work done by the fluid on the surroundings is ⊕ ive

q → energy absorbed by the system — ⊕ ive
W → work done by the system — ⊕ ive

Thus. We have, overall energy balance =

$$\iint_A \left(H + \frac{v^2}{2} + zg\right) \rho v \cos\alpha \, dA + \frac{\partial}{\partial t} \iiint_V \left(U + \frac{v^2}{2} + zg\right)_{\rho dv}$$

$$= q - \dot{W}_s \quad \text{—— (A)}$$

↳ $\dot{W}_s$ work done by the fluid.

Overall Energy balance and steady state flow

~~$\rho_1 = (m_1/A_1 v_1)$, $\rho_2 = m_2/(A_2 v_2)$ then eqn~~

From eqn (A) substituting becomes

$$H_2 m_2 - H_1 m_1 + \frac{m_2 (v_2^3)_{av}}{2 v_{2av}} - \frac{m_1 (v_1^3)_{av}}{2 v_{1av}} + (g m_2 z_2 - g m_1 z_1) = q - \dot{w}_s$$

↳ based on per unit time

for steady flow $m_2 = m_1 = m$

$$H_2 - H_1 + \frac{1}{2\alpha}\left(v_{2av}^2 - v_{1av}^2\right) + (z_2 - z_1) g = Q - W_s \quad \text{—— (1)}$$

$\alpha \to$ kinetic-energy velocity correction factor

for various flow in pipes

$\alpha_{laminar} \to \frac{1}{2}$

$\alpha_{turbulent} \to 1.0$

Kinetic-Energy velocity correction factor, α

17844

−271.055 KJ

$$K.E. = \iint_A \left(\frac{v^2}{2}\right) \rho v \cos\alpha \, dA \quad \text{— let } \rho \to \text{constant}$$

$$= \frac{\rho}{2} \iint_A v^3 \cos\alpha \, dA$$

$$= \frac{\rho v_{av} \cdot A}{2 v_{av} \cdot A} \iint_A v^3 \cos\alpha \, dA$$

$$= \frac{m}{2 v_{av}} \cdot \left[\frac{1}{A} \iint v^3 \cos\alpha \, dA\right] \quad \to \langle v^3 \rangle_{avg}$$

↳ 1.0

$\alpha = 0$ normal surface

$$\frac{K.E}{m} = \frac{1}{2}\cdot\frac{1}{V_{av}}\cdot\left[\frac{1}{A}\iint v^3\,dA\right] \rightarrow (v^3)_{av}$$

~~$= \frac{V_{av}^3}{2V_{av}} = \frac{V_{av}^3}{2\alpha}$~~

$$= \frac{(V^3)_{av}}{2\,V_{av}}$$

$$\Rightarrow \frac{(V^3)_{av}\cdot(V_{av})^2}{2V_{av}(V_{av})^2} = \frac{V^2_{av}}{2\alpha} \quad \text{where} \quad \alpha = \frac{V_{av}^3}{V^3_{av}}$$

★ for laminar flow $\alpha = 0.5$

Use $v = 2V_{av}\left[1 - \left(\frac{r}{R}\right)^2\right]$

where $A = \pi R^2$ & $dA = r\,dr\,d\theta$

> in cartesian coordinates $dA = dx.dy$ but in polar coordinates (for pipe)
> $dA = r\,dr.d\theta$
> $A = \pi R^2$

★ for turbulent flow

$$v = V_{max}\left(\frac{R-r}{R}\right)^{1/7}$$

α varies $\rightarrow$ 0.90 — 0.99

$\alpha \simeq 1.0$ for turbulent flow

Note: If there is significant change in the Enthalpy or appreciable heat is added or removed P.E. & K.E. terms can be safely neglected from that eqn

See example 2.7-1 (Geankoplis)
-2
-3

Overall Mechanical Energy Balance eqn.

(We are more concerned with the Mechanical Energy)

Mechanical Energy is a form of energy that can be converted into work.

Energy converted to heat / internal energy is lost work or a loss in mechanical energy which is caused by frictional resistance to flow. For s.s. flow when a unit mass of fluid passes from inlet to outlet, the batch work done by the fluid, W' is

$$W' = \int_{V_1}^{V_2} p\,dV - \sum F \quad (\sum F > 0)$$

$\sum F$ ↳ sum of all frictional losses / mass of fluid

Pay more attention for explaining this topic

from first law of thermodynamics

$$\Delta U = Q - W'$$

Note: $\sum F$ is actually a loss of ~~work~~ in Mechanical Energy due to frictional resistance to flow.

Note:
W' is different from W used previously as the later also includes the K.E and P.E. effects.

Enthalpy ~~balance~~ equation is defined as

$$\Delta H = \Delta U + \Delta PV$$

or

$$\Delta H = \Delta U + \int_{V_1}^{V_2} p\,dv + \int_{P_1}^{P_2} v\,dp$$

$\rightarrow W' + \Sigma F$

$$\therefore \Delta H = Q - W' + W' + \Sigma F + \int_{P_1}^{P_2} v\,dp$$

$$\boxed{\Delta H = Q + \Sigma F + \int_{P_1}^{P_2} v\,dP}$$

→ substituting this eqn into S.S. overall energy balance eqn

$$\frac{1}{2\alpha}\left[v_{2\,av}^2 - v_{1\,av}^2\right] + g(z_2 - z_1) + \int_{P_1}^{P_2} \frac{dP}{\rho} + \Sigma F + W_s = 0$$

which is the desired overall

overall mechanical

Energy balance eqn

$$\boxed{v = 1/\rho}$$

for incompressible fluid

$$\int_{P_1}^{P_2} \frac{dP}{\rho} = \frac{P_2 - P_1}{\rho}$$

and the eqn becomes

$$\frac{1}{2\alpha}\left[v_{2\,av}^2 - v_{1\,av}^2\right] + g(z_2 - z_1) + \frac{P_2 - P_1}{\rho} + \Sigma F + W_s = 0$$

Solve ex 2.7-4 / 2.7-5

Bernoulli Equation for Mechanical Energy Balance:

$W_s = 0$, $\Sigma F = 0$ } Bernoulli's case

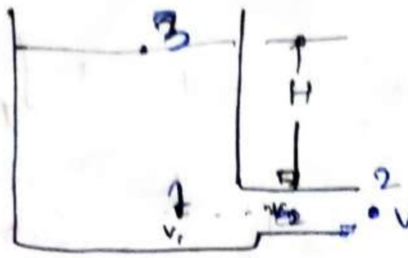
no mechanical energy is added ← $W_s = 0$

$\Sigma F = 0$ → no frictional losses

then for turbulent flow

$$\boxed{z_1 g + \frac{v_{1\,av}^2}{2} + \frac{P_1}{\rho} = z_2 g + \frac{v_{2\,av}^2}{2} + \frac{P_2}{\rho}}$$

Flow from a nozzle in a tank:



at pt 2 pressure is same as pt. 3

$P_3 = P_2$.

from Bernoullis eqn

$Z_1 = Z_2$, ∴ $\alpha = 1.0$ (turbulent flow)

$$\frac{V_1^2}{2} + \frac{P_1 - P_2}{\rho} = \frac{V_2^2}{2}$$

$$\boxed{P_1 - P_2 = \rho g H}$$ → because at pt. 2 the pressure is same as that of pt '3'

∴ $V_1 = 0$, ∵ tank is very large compared

$$\boxed{V_2 = \sqrt{2gH}}$$

(Q) 2.7-6 / 72-Geankoplis | A liquid with a constant density ρ kg/m³ is flowing at an unknown velocity v_1 m/s through a horizontal pipe of cross-sectional Area A_1 m² at a pressure p_1 N/m², and then it passes to a section of the pipe in which the area is reduced gradually to A_2 m² and the pressure is p_2. Assuming no friction losses, calculate velocity v_1 and v_2 if the pressure difference ($p_1 - p_2$) is measured.

Soln $\rho_1 = \rho_2 = \rho$ ∴ $v_2 = \frac{v_1 A_1}{A_2}$ continuity eqn

$Z_1 = Z_2 = 0$ for horizontal pipes

Now from Bernoullis eqn

$$0 + \frac{v_1^2}{2} + \frac{p_1}{\rho} = 0 + \frac{v_1^2 A_1^2 / A_2^2}{2} + \frac{p_2}{\rho}$$

$$v_1 = \sqrt{\frac{(p_1 - p_2)}{\rho} \cdot \frac{2}{\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]}}$$

$$v_2 = \frac{v_1 A_1}{A_2}$$

Overall Momentum Balance:

momentum → vector

mass, Energy → Scalar

Newton's second law

$$\sum F = \frac{d}{dt}(m \cdot \bar{v}) = \frac{d}{dt}\bar{P}$$

$\bar{P}$ → Total linear momentum of the system.

Equation for the conservation of momentum w.r.t. to C.V.

(Sum of forces acting on C.V.) = (rate of momentum out of control vol.) − (rate of momentum into C.V.) + rate of acc. of momentum in control volume.

↳ Same as regeneration term.

* If external forces are absent ~~it is~~ momentum is ~~not~~ conserved, otherwise it is not conserved as it is generated by the external force.

The first two terms on R.H.S gives the (net) rate of momentum efflux.

For a small elemental area dA ~~on~~ the C.V. surface

Rate of momentum efflux = $v(\rho v)(dA \cos\alpha)$ ~~$v(\rho v)(dA\cos\alpha)$~~

$$= \rho\bar{v}\,(\bar{v}\cdot\bar{n})\,dA$$

× net momentum efflux from C.V. $= \iint_A \bar{v}\,(\rho \cdot v)\cos\alpha\, dA = \iint_A \rho\bar{v}\,(\bar{v}\cdot\bar{n})\,dA$

rate of accumulation of momentum in c.v.

$$= \frac{\partial}{\partial t} \iiint_V \rho \bar{v} \, dV$$

$$\therefore \sum F = \iint_A \rho \bar{v} (\bar{v} \cdot \bar{n}) \, dA + \frac{\partial}{\partial t} \iiint_V \rho \bar{v} \, dV$$

$\sum F$ may component in any direction
say x direction

$$\sum F_x = \iint_A \rho \bar{v}_x (\bar{v} \cdot \bar{n}) \, dA + \frac{\partial}{\partial t} \iiint_V \rho \bar{v}_x \, dV$$

$$= \iint_A \bar{v}_x \, \rho \bar{v} \cos\alpha \, dA + \frac{\partial}{\partial t} \iiint_V \rho \bar{v}_x \, dV$$

So on

$\sum F_x$ (like reaction force) →
- Body force (gravitational force) - F_{xg}
- Surface force (~~shear force~~) - R_x
- Pressure force — F_{xp}
- Friction force — F_{xs}

Surface force :- In case where the control surface cuts through a solid, there is present force R_x, (reaction)

$$\sum F_x = F_{xg} + F_{xs} + F_{xp} + R_x$$

$$= \iint_A v_x (\rho v) \cdot \cos\alpha \, dA + \frac{\partial}{\partial t} \iiint_V \rho v_x \, dV$$

Overall momentum balance in flow system in one direction:-

Consider the steady state flow

$$\sum F_x = F_{xg} + F_{xp} + F_{xs} + R_x = \iint_A v_x (\rho v_x) \cos\alpha \, dA$$

$\therefore v = v_x$

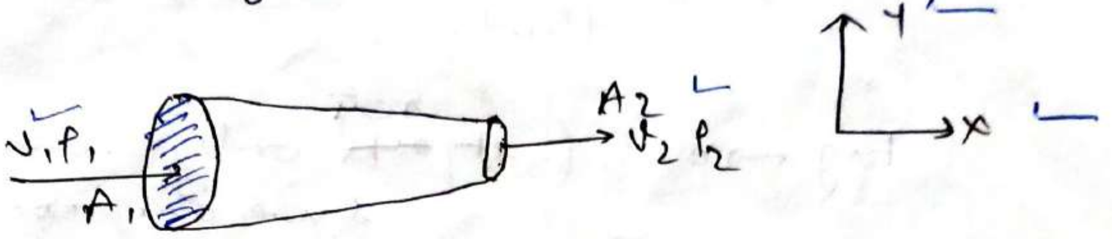
$x \longrightarrow$

Flow in x direction   let $\cos\alpha = \pm 1, 0$ (33)

Now $\rho A = \frac{m}{V_{av}}$ $\rho = \frac{m}{A \cdot V_{av}}$

$$\therefore \Sigma F = m \frac{(V_{x2})^2_{av}}{V_{x2\,av}} - m \frac{(V_{x1})^2_{av}}{V_{x1\,av}} \quad (1)$$

if the velocity is not constant and varies across the surface



in that case

$$(V_x^2)_{av} = \frac{1}{A} \iint_A V_x^2 \cdot dA$$

define momentum velocity correction factor.

$$\frac{V_{x\,av}}{\beta} = \frac{(V_x^2)_{av}}{V_{x\,av}} \quad \text{or} \quad \boxed{\beta = \frac{(V_{x\,av})^2}{(V_x^2)_{av}}}$$

$\beta \longrightarrow 0.95 - 0.99$ for turbulent flow

$\longrightarrow$ $3/4$ $\left(\frac{4}{3}\right)$? for laminar flow

~~in general~~

For turbulent flow we can safely right $\frac{(V_x)^2_{av}}{V_{x\,av}} = (V_{x\,av}) = (V)$

$\beta \to 1$ as we are considering only one direction

dropping x for one dimension flow
$V_x = V$ & $F_x = F$

$\therefore \sum F = m\left(\frac{v_2}{\beta}\right) - m\left(\frac{v_1}{\beta}\right)$ | From eq^n 1

$\sum F = F_{xg} + F_{xs} + F_{xp} + R_x$

$R_x \rightarrow$ force exerted by the solid on the fluid

$F_{xg} \rightarrow 0$ (acts only in y direction & we are considering x direction.

$F_{xs} \rightarrow$ (friction force) usually very small

Rel. $\beta \rightarrow 1.0$ (consider)

$$F_{xp} + R_x = m v_2 - m v_1$$

$F_{xp} = P_1 A_1 - P_2 A_2$ Where Fxp is the force caused by pressure acting over control volume.

$$\therefore R_x = m v_2 - m v_1 + P_2 A_2 - P_1 A_1$$

if fluid exerts pressure on the solid. (reaction force) R_x has ⊖ve sign.

Momentum Velocity correction factor β for Laminar flow:

$$\frac{(V^2)_{av}}{V_{av}} = \frac{V_{av}}{\beta}$$

$$\therefore \beta = \frac{(V_{av})^2}{(V^2)_{av}}$$

Let's find β for laminar flow ~~Now for laminar flow~~

$$\beta = (V^2)_{av} = \frac{1}{A}\iint_A v^2\, dA$$

$V_{max} = 2V_{av}$

$A = \pi R^2$, $dA = r\,dr\,d\theta$

$$(V^2)_{av} = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R \left[2V_{av}\left(1 - \frac{r^2}{R^2}\right)\right]^2 r\,dr\,d\theta$$

$$= \frac{(2\pi)2^2 V_{av}^2}{\pi R^2}\int_0^R \frac{(R^2 - r^2)^2}{R^4}\, r\,dr$$

Note for laminar flow

$$v = V_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

also $V_{av} = \frac{V_{max}}{2}$

$$(V^2)_{av} = \frac{8V_{av}^2}{R^6}\left(\frac{R^6}{2} - \frac{R^6}{2} + \frac{R^6}{6}\right)$$

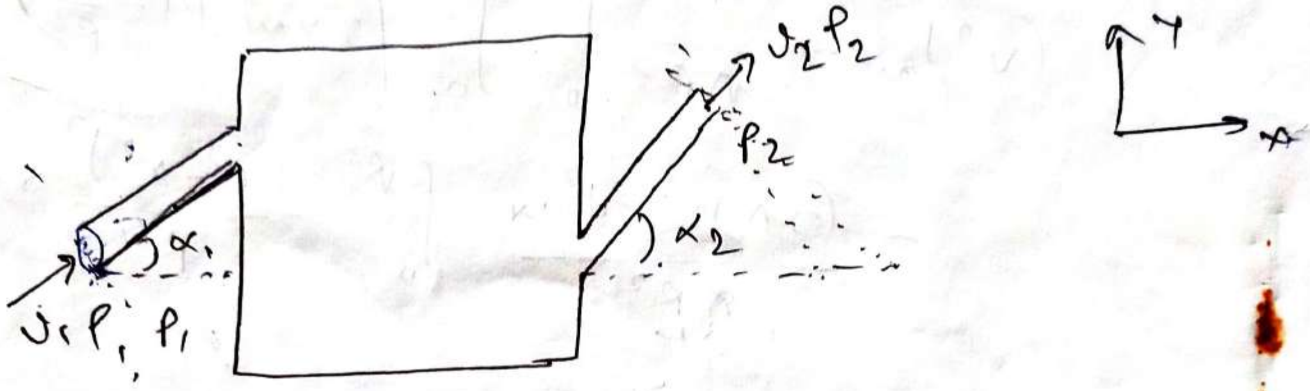
$$= \frac{4}{3} V_{av}^2$$

$$\beta = \frac{(V_{av})^2}{(V^2)_{av}} = \frac{3}{4}$$

$$\Rightarrow \frac{3}{4} = \frac{(V_{av})^2}{(V^2)_{av}} = \beta$$

Overall momentum balance in two direction

Application of momentum balance eqn



Assume S.S flow & no friction force i.e $F_{xs} = 0$

$$\therefore F_{xg} + F_{xp} + R_x = \iint_A v(\rho \cdot v) \cos\alpha \, dA$$

(F_{xg} → 0 for x direction)

$\therefore \rho A = \frac{m}{v_{av}}$

$\rho = \frac{m}{A v_{av}}$

$$= \frac{m (V_2^2)_{av}}{V_{2\,av}} \cos\alpha_2 - \frac{m (V_1^2)_{av}}{V_{1\,av}} \cos\alpha_1$$

$F_{xp} = P_1 A_1 \cos\alpha_1 - P_2 A_2 \cos\alpha_2$

$R_x = P_2 A_2 \cos\alpha_2 - P_1 A_1 \cos\alpha_1 + m v_2 \cos\alpha_2 - m v_1 \cos\alpha_1$

$$\frac{V^2_{av}}{(V_{av})} = \frac{V_{av}}{\beta}$$

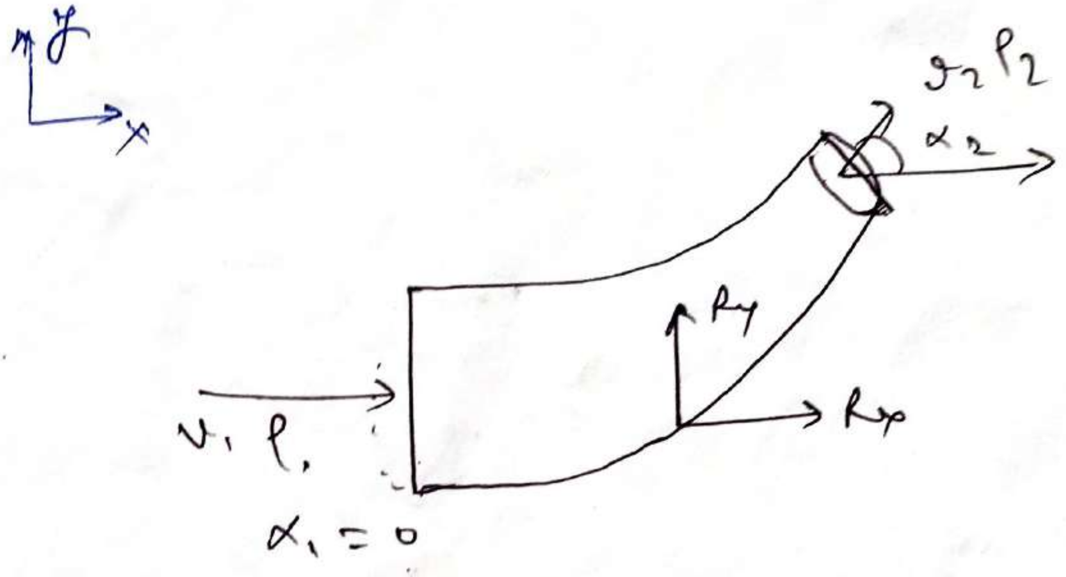
$\beta = 1.0$

For y direction

$$R_y = m v_2 \sin\alpha_2 - m v_1 \sin\alpha_1 + P_2 A_2 \sin\alpha_2 - P_1 A_1 \sin\alpha_1 + m_t g$$

$m_t \rightarrow$ total mass of fluid within C.V. | $F_{yg} = - m_t g$ acting ↓

Flow Through a pipe bend



$$R_x = m v_2 \cos\alpha_2 - m v_1 + P_2 A_2 \cos\alpha_2 - P_1 A_1$$

$$R_y = m v_2 \sin\alpha_2 - 0 + P_2 A_2 \sin\alpha_2 - 0 + mg$$

$$R_y = m v_2 \sin\alpha_2 - P_2 A_2 \sin\alpha_2 + mg$$

$$|R| = \sqrt{R_x^2 + R_y^2}$$

magnitude of resultant force.

$$\theta = \arctan\left(\frac{R_y}{R_x}\right)$$

UNIT-II

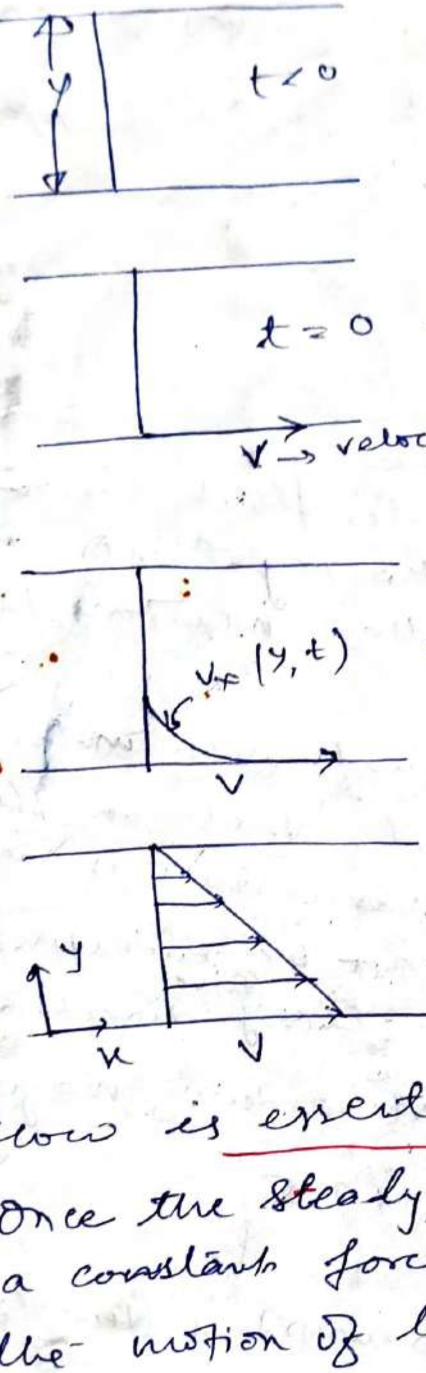
Transport Phenomena

(37-a)

Unit - II

Newton's law of viscosity.

Consider the flow between two large plates of area 'A'



t<0 : fluid initially at rest

t = 0 : lower plate set in motion with velocity 'V'

small t : velocity build up in unsteady flow

Large t : Final velocity distribution in steady flow

flow is essentially laminar

Once the steady state has been attained a constant force 'F' will be required to maintain the motion of the lower plate.

thus we can write as

$$\boxed{\frac{F}{A} = \mu \frac{V}{Y}}$$

μ → constant of proportionality called viscosity

For antiproportionality of Y consider the example of two slide b/w which a thin layer of fluid is placed. Then it requires larger force to compared to when the fluid thickness b/w plate is more.

(37b)

let

$\frac{F}{A} = \tau_{yx} \rightarrow$ force in x direction on a unit area ⊥ to 'y' direction

It is understood that this force is exerted by the fluid of lesser y on the fluid of greater 'y'. as such replace the $\frac{V}{Y} = -\frac{dv_x}{dy}$

Therefore

$$\boxed{\tau_{yx} = -\mu \frac{dv_x}{dy}}$$

Above eqn which states that shearing force per unit area is ∝ to the negative of the velocity gradient is called Newtons law of viscosity."

It is applicable for liquids with molecular weight < 5000, called 'Newtonian fluid' ↳ (μ is constant at constant Temp)

(Explain this) Alternately above eqn can be interpreted as flux of x momentum transferred in +ive direction

(any unit notation) Fluid viscosity varies over order of magnitude

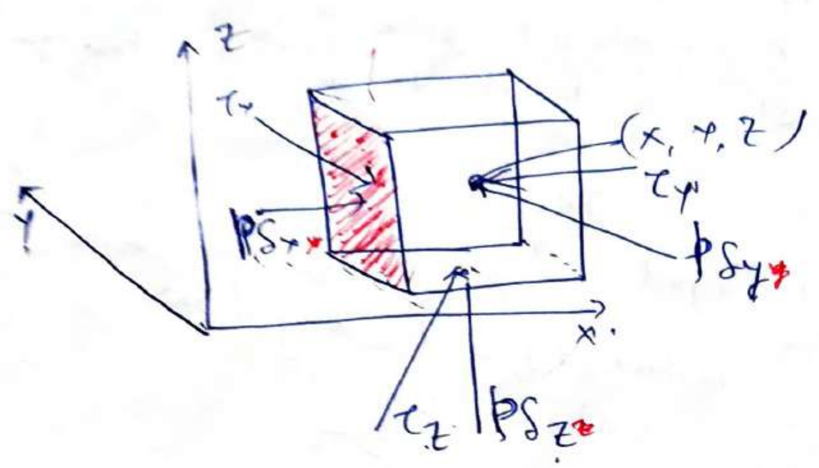
for air - 20°C - 1.8×10^{-5} Pa.s.

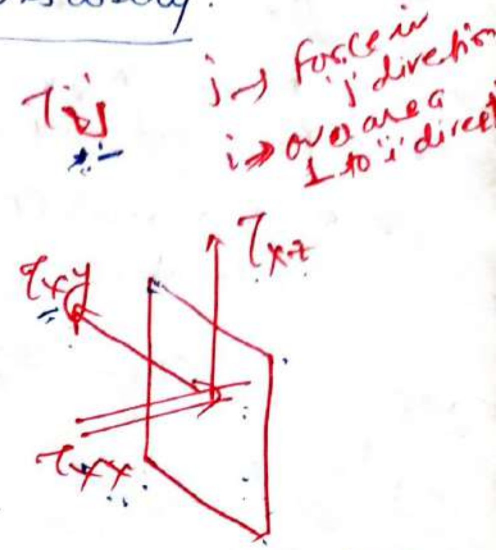
glycerol — 1 Pa.s.

in Gases momentum is transported due to free molecular collision

liquids → momentum is transferred due to intermolecular forces that pairs of molecules experience as they interact with their neighbours.

Generalization of Newtons law of viscosity.





Consider a general flow pattern such that

$V_x = V_x(x, y, z, t)$

$V_y = V_y(x, y, z, t)$

$V_z = V_z(x, y, z, t)$

Pressure force will always be ⊥ to the exposed surface. $p\delta_x$, $p\delta_y$ & $p\delta_z$ are pressure forces in the x, y & z direction respectively. δ_i → is a unit vector.

Viscous forces (τ_x, τ_y, τ_z) come into play when their are velocity gradient within the fluid. In general they are neither ⊥ nor ∥ to it.

Each of the viscous forces τ_x, τ_y & τ_z have components like τ_{xx}, τ_{xy} & τ_{xz}. Sum of the forces acting on the three faces shown above can be written as

$$\boxed{\pi_{ij} = p\delta_{ij} + \tau_{ij}}$$

Tensor ← π_{ij}; Tensor ← δ_{ij}; τ_{ij} → Tensor

π_{ij} → molecular stress

δ_{ij} = Kronecker delta

when $i = j$, $\delta_{ij} = 1$

$i \neq j$, $\delta_{ij} = 0$

11 21 31
12 22 32
13 23 33

i & j may be x, y or z

π → has '9' components

(37d)

$\tau_{xx}, \tau_{yy}, \tau_{zz}$ are called normal stresses

$\tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}$ .... are called shear stress

to avoid confusion $\pi \rightarrow$ molecular stress tensor
$\tau \rightarrow$ viscous stress tensor.

- How are these stresses related to velocity gradient.

$$\tau_{ij} = -\sum_k \sum_l \mu_{ijkl} \frac{\partial v_k}{\partial x_l}$$

where i, j, k, l, may have values 1, 2, 3

$\mu_{ijkl} \rightarrow$ have 81 quantities $\rightarrow (3^4)$

Restrictions in generalization

- If the fluid is under pure rotation no viscous force is present. This means $\tau_{ij} = \tau_{ji}$ ~~be a symmetric combination of the velocity gradients.~~ That means if i & j are interchanged, the combination of velocity gradients remains unchanged. It can be shown that the only symmetric linear combinations of velocity gradients are

$$\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right) \quad \& \quad \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\delta_{ij}$$

- Consider the fluid is isotropic

$$\tau_{ij} = A\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + B\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\delta_{ij}$$

We have thus reduced the number of viscosity coefficients from 81 to 2!

37e

For the simple flow considered here theory suggest that

$A = -\mu$ & $B = \left(\frac{2}{3}\mu - k\right)$

$k \rightarrow$ dilatational viscosity

$= 0$ for monoatomic gas at lower density (ideal gas)

$$\tau_{ij} = \tau_{ji} = -\mu\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right) + \left(\frac{2}{3}\mu - k\right)\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\delta_{ij}$$

$i = 1, 2, 3$ & $j = 1, 2, 3$

for incompressible fluid $\nabla \cdot v = 0$

if the fluid is incompressible

$$(\nabla \cdot v) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

(from continuity eqn we'll see that)

Alternatively general form

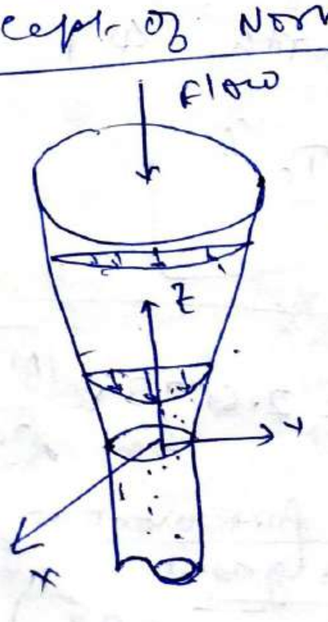
$$\underline{\tau} = -\mu\left(\nabla v + (\nabla v)^T\right) + \left(\frac{2}{3}\mu - k\right)(\nabla \cdot v)\underline{\delta}$$

∇v → velocity gradient

μ → scalar

$(\nabla \cdot v)$ → divergence of velocity vector

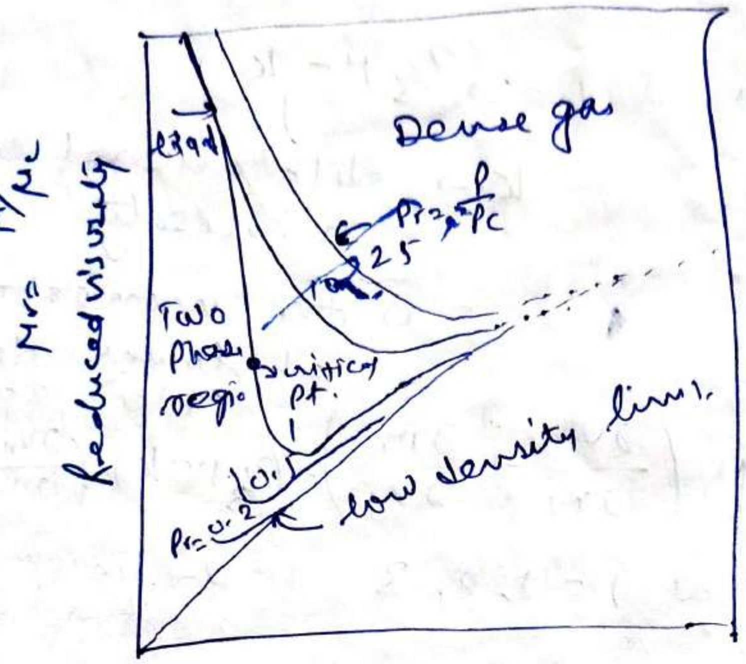
Concept of Normal Stresses



v is fn. of (r, z) ∴ $\tau_{zz} = -2\mu\left(\frac{\partial v_z}{\partial z}\right)$

$\tau_{rr} = -2\mu\left(\frac{\partial v_r}{\partial r}\right)$

Pressure & Temperature dependence of μ:



The chart shows that viscosity of a gas approaches a limit (the low density limit) as the pressure becomes smaller, for most gases this limit is ~~nearly~~ nearly attained at 1 atm.

μ gas at low density ↑ with ↑T

μ liquid ↓ with ↑T

$$\mu_c = 7.70\, M^{1/2}\, P_c^{2/3}\, T_c^{1/6}$$

ex. 1.3-1

refer to eqn 1.4-14 $\mu = 2.6693 \times \frac{\sqrt{MT}}{\sigma^2 \Omega_\mu}$

pure monoatomic for gases

exp. 1.4-1 liquid $\mu = A \exp(B/T)$

UNIT- II - (BSL)

Shell Momentum Balance & Velocity distribution in Laminar Flow

Shell Momentum balances & B.C.

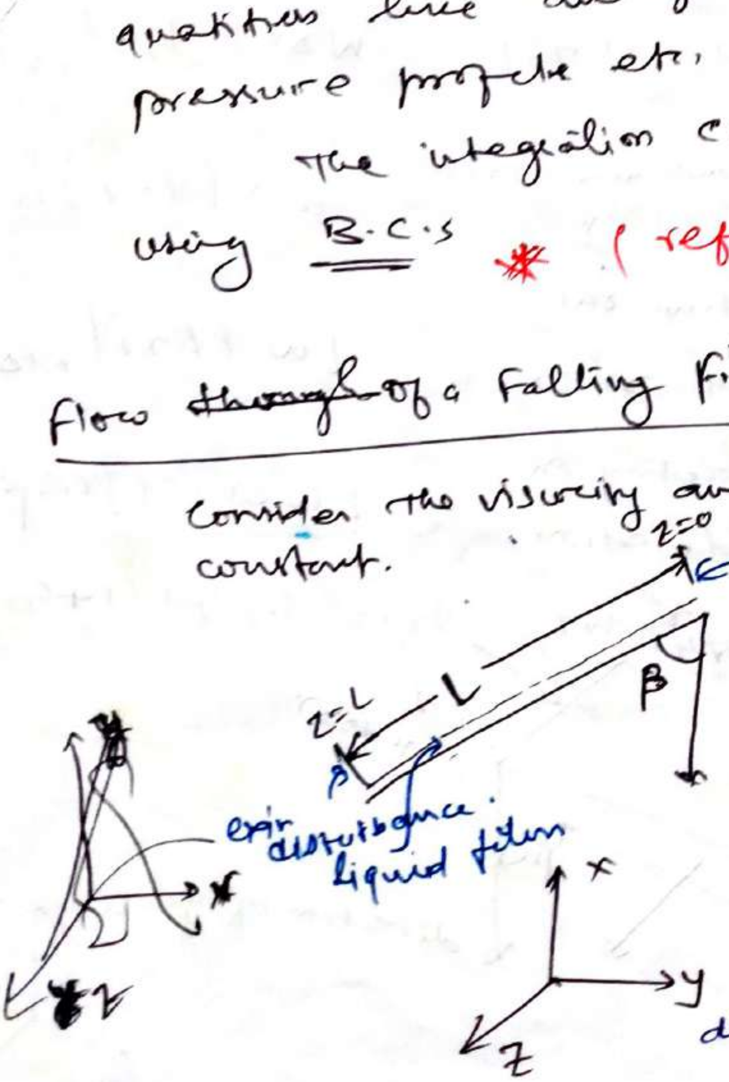
We have seen the overall balance eqn but that ~~does~~ did not ~~tell us about the~~ give insight of control volume. Here we address this problem.

Here we consider a small c.v. then shrink it further to get a differential eqn. this diff. eqn is then integrated to get quantities like average velocity, velocity profile, pressure profile etc.

The integration constants are evaluated using B.C.s * (refer to supplement page.)

Flow ~~through~~ of a Falling Film:

Consider the viscosity and density of the fluid are constant.

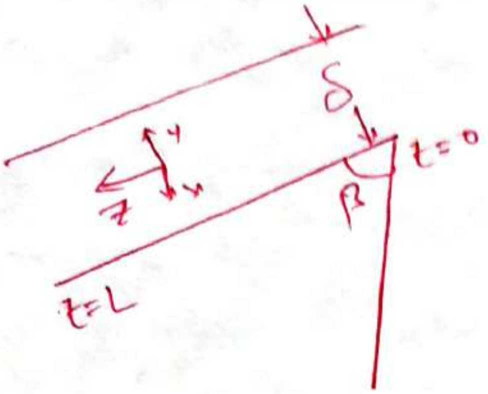


$z=0,\ z=L$ | edges
$y=0,\ y=W$ |

$W \rightarrow$ width of the plate

Consider W & L are very large compared to δ, as such the ~~disturb.~~ disturbance at the entry and exit edges can be neglected

How to select shell?



Velocity vector $\underline{V}$ has components

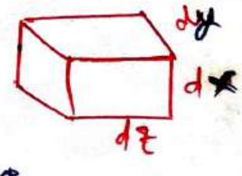
V_z, V_x, V_y

But $V_x = V_y = 0$
for this problem & $V_z \neq 0$

$V_x \to$ x momentum ($\because V_x = 0$)
$V_y \to$ y momentum ($V_y = 0$)
$V_z \to$ z momentum : ($V_z \neq 0$)

Now $V_z = f(x)$ only therefore momentum is transferred in the x direction therefore select an area of thickness $\underline{dx}$ in ⊥ direction to $\underline{x}$.

We can choose shell of



this size

but the integration will be complicated. and further using dx & dy is of no use as there is no momentum transport in x, & y direction

* If the velocity is not changing in a direction the shell may cover the whole system in that direction.

* For small flow rate, prevent viscous force will acceleration of the fluid down the wall. therefore $v_z \neq (z)$

Summary of Notation for Momentum Fluxes

Symbol	Meaning	Reference
$\rho v v$	convective momentum flux-Tensor	Table 1.7-1
τ	viscous momentum flux Tensor	
$\pi = p\delta + \tau$	Molecular momentum-flux Tensor	
$\phi = \pi + \rho v v$	Combined momentum flux Tensor	

It should be noted that $v_x = v_y = 0$ & $P = P(x)$

Now let's consider a fluid system ⊥ to x direction and make a shell balance over (Δx)

rate of z-momentum in across surface of z=0 — $(W\Delta x)\, \phi_{zz}|_{z=0}$

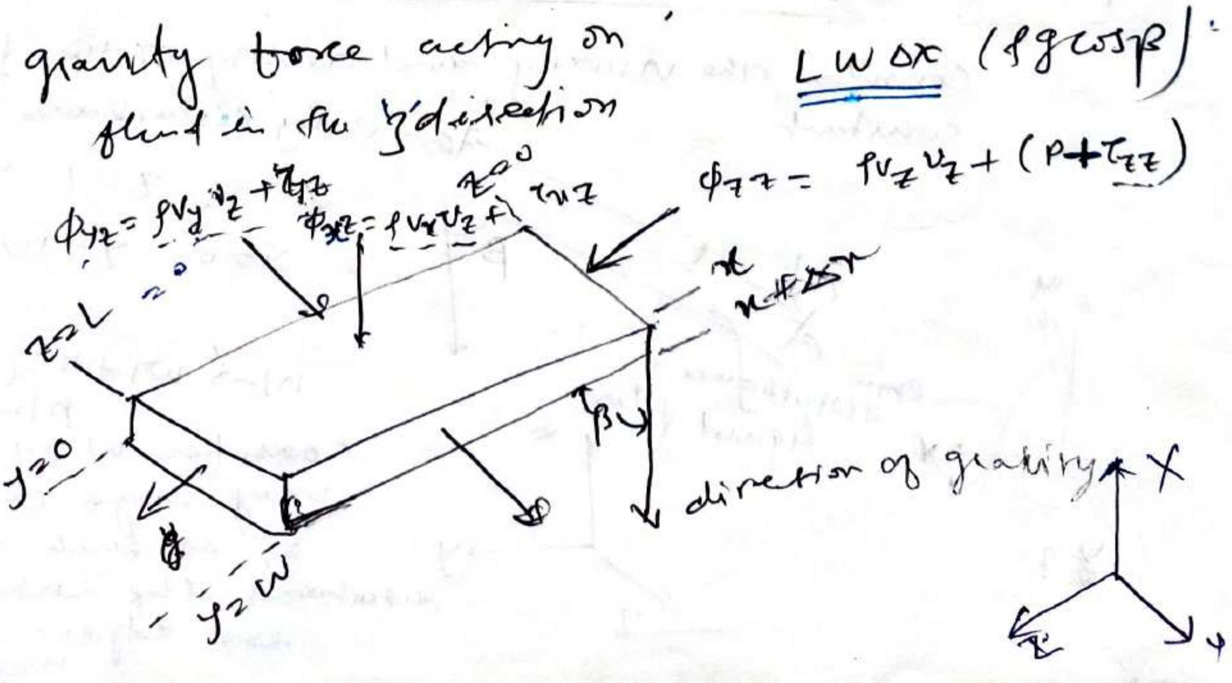
— " — out at z=L — $(W\Delta x)\, \phi_{zz}|_{z=L}$

rate of z momentum in across surface at x — $LW(\phi_{xz})|_x$

Rate of z momentum out across surface of x+Δx — $LW(\phi_{xz})|_{x+\Delta x}$

gravity force acting on fluid in the z direction — $LW\Delta x\,(\rho g \cos\beta)$

$\phi_{zz} = \rho v_z v_z + (p + \tau_{zz})$



$\underline{\underline{\phi}} = \underline{\underline{\pi}} + \rho \underline{v}\,\underline{v}$ where $\underline{\underline{\pi}} = p\underline{\underline{\delta}} + \underline{\underline{\tau}}$

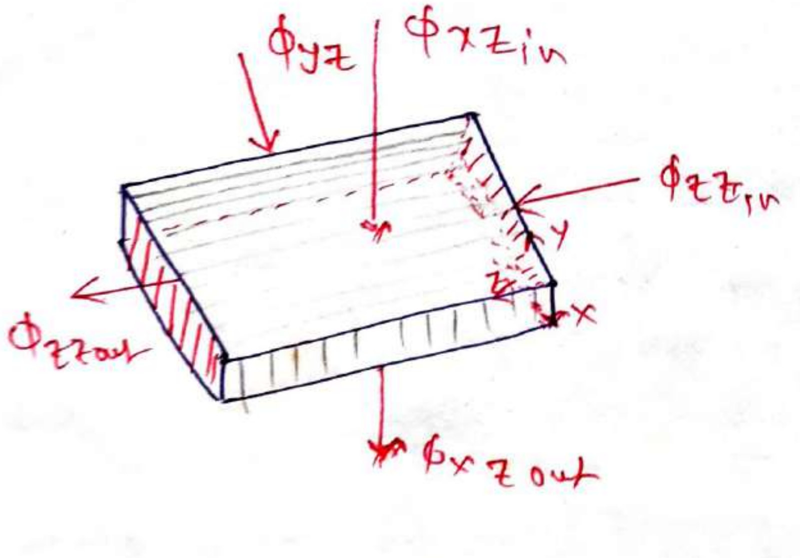
This $\underline{\phi}$ will have nine components

$$\underline{\underline{\phi}} = \begin{bmatrix} \phi_{xx} & \phi_{xy} & \phi_{xz} \\ \phi_{yx} & \phi_{yy} & \phi_{yz} \\ \phi_{zx} & \phi_{zy} & \phi_{zz} \end{bmatrix} = p\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} + \begin{bmatrix} \rho v_x v_x & \rho v_x v_y & \rho v_x v_z \\ \rho v_y v_x & \rho v_y v_y & \rho v_y v_z \\ \rho v_z v_x & \rho v_z v_y & \rho v_z v_z \end{bmatrix}$$

In this problem $v_x = v_y = 0$ ∴ no momentum in x & y direction therefore only existing term will be

$$\underline{\underline{\phi}} = \boxed{\phi_{xz}} \quad \phi_{yz} \quad \boxed{\phi_{zz}}$$

second subscript → flow direction
first → Area ⊥ to that direction.



v_x & v_y are both zero, $\rho v_x v_z$ and $\rho v_y v_z$ are zero.. Since v_z does not depend on y and z, it follows ~~from Table B~~ that $\tau_{yz} = 0$ & $\tau_{zz} = 0$. Therefore the terms with dashed lines 'shown in the pg' do not appear in the final momentum balance equation for the balance of z momentum.

$$LW(\phi_{xz}|_x - \phi_{xz}|_{x+\Delta x}) + W\Delta x(\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L})$$
$$+ LW\Delta x\,(\rho g \cos\beta) = 0 \quad -(1)$$

Divide the eqn by $LW\Delta x$ and take limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0}\left(\frac{\phi_{xz}|_{x+\Delta x} - \phi_{xz}|_x}{\Delta x}\right) - \frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} = \rho g \cos\beta$$

$$\frac{\partial \phi_{xz}}{\partial x} - \frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} = \rho g \cos\beta.$$

Now

$$\phi_{xz} = \rho v_x v_z + \tau_{xz} = -\mu \frac{\partial v_z}{\partial x} + \rho v_x v_z$$

$$\phi_{zz} = p + \tau_{zz} + \rho v_z v_z = p - 2\mu \frac{\partial v_z}{\partial z} + \rho v_z v_z$$

(look in to the generalized eqn of ~~viscosity~~ Newton's law of viscosity eqn 1.2-6 pp 18

(41)

$v_x = v_y = 0$, $P = P(x)$ also $\frac{\partial v_z}{\partial z} = 0$

as $v_z = v_z(x)$

$$\frac{\partial \phi_{xz}}{\partial x} - \frac{\phi_{zz}|_{z=0} - \phi_{zz}|_{z=L}}{L} = \rho g \cos\beta$$

$$\frac{\partial \tau_{xz}}{\partial x} - 0 = \rho g \cos\beta$$

here note that $\phi_{zz} = P - 2\mu \frac{\partial v_z}{\partial z} + \rho v_z v_z$ ($\frac{\partial v_z}{\partial z} = 0$)

therefore $P_{in} = P_{out}$ & $\rho v_z v_z{}_{in} = \rho v_z v_z{}_{out}$

$$\boxed{\frac{d\tau_{xz}}{dx} = \rho g \cos\beta}$$

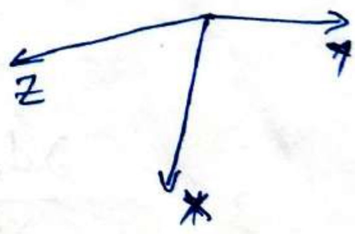
As we consider that momentum is transferred from lesser 'x' to greater 'x' or the force exerted by fluid of lesser 'x' on the fluid of larger 'x'

integrating at $x = 0$, $\tau_{xz} = 0$, B.C.

$\tau_{xz} = \rho g \cos\beta \, x$

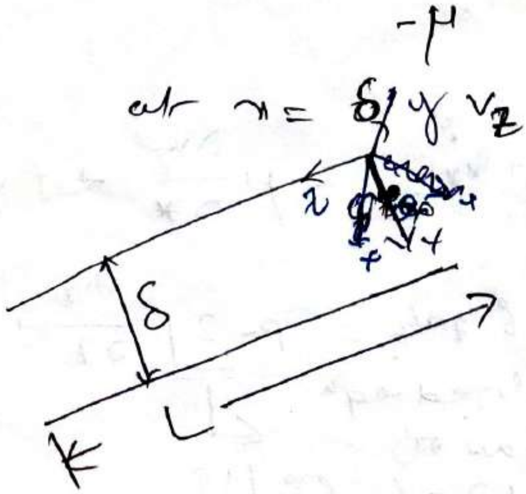
$\tau_{xz} = -\mu \frac{dv_z}{dx}$ (Newton's law)

$\therefore -\mu \frac{dv_z}{dx} = \rho g \cos\beta \, x$



$$dv_z = \frac{\rho g \cos\beta}{-\mu} x \cdot dx$$

at $x = \delta$, $v_z = 0$ (No slip Boundary condition)



$$V_z = \frac{\rho g \cos\beta\, \delta^2}{2\mu}\left[1-\left(\frac{x}{\delta}\right)^2\right]$$

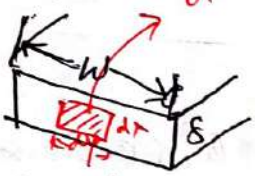
Parabolic velocity distribution

(i) <u>Max velocity</u> ? $x = 0$, $V_z = V_{z\,max}$

$$V_{z,max} = \frac{\rho g \cos\beta\, \delta^2}{2\mu}$$

$dx \cdot dy$ (area ⊥ to the direction of flow

(ii) avg velocity



$$\langle V_z\rangle = \frac{\int_0^W \int_0^\delta V_z\, dx\, dy}{\int_0^W \int_0^\delta dx\, dy} = \frac{W\int_0^\delta V_z\, dx}{W\delta}$$

$$= \frac{\int_0^\delta V_z \cdot dx}{\delta} = \frac{\frac{\rho g\cos\beta}{2\mu}\int_0^\delta \left[\delta^2 - x^2\right] dx}{\delta}$$

$$= \frac{\rho g \cos\beta}{2\mu\,\delta}\left[\delta^2 x - \frac{x^3}{3}\right]_0^\delta = \frac{\rho g\cos\beta}{2\mu\delta}\cdot\delta^3\left[1-\frac{1}{3}\right]$$

$$= \frac{\rho g \cos\beta \cdot \delta^2}{3\mu} = \frac{2}{3} V_{z\,max}$$

$$\langle V_z\rangle = \frac{2}{3} V_{z\,max}$$

(43)

(iii) Mass flow rate

$$w = \int_0^W \int_0^\delta \rho\, v_z\, dx\, dy$$

Area ⊥ to the flow direction

$$= \rho W \int_0^\delta v_z\, dx = \rho W \cdot \delta \left[\frac{\int_0^\delta v_z\, dx}{\delta} \right]$$

$$= \rho W \cdot \delta \cdot \langle v_z \rangle$$

$$\dot{w} = \frac{\rho^2 g W \delta^3 \cos\beta}{3\mu}$$

(iv) **Film thickness 'δ'**

~~$\delta = \frac{\rho^2 g w}{}$~~

$$\delta = \left(\frac{3\mu \dot{w}}{\rho^2 g \cos\beta} \right)^{1/3}$$

(v) The force per unit area in $z \rightarrow$ on a plane ⊥ x direction $= \tau_{xz}$ at $x = \delta$

this force ~~ext~~ exerted by the fluid on the wall

$$F = \int_0^L \int_0^W \left(\tau_{xz} \big|_{x=\delta} \right) dy \cdot dz$$

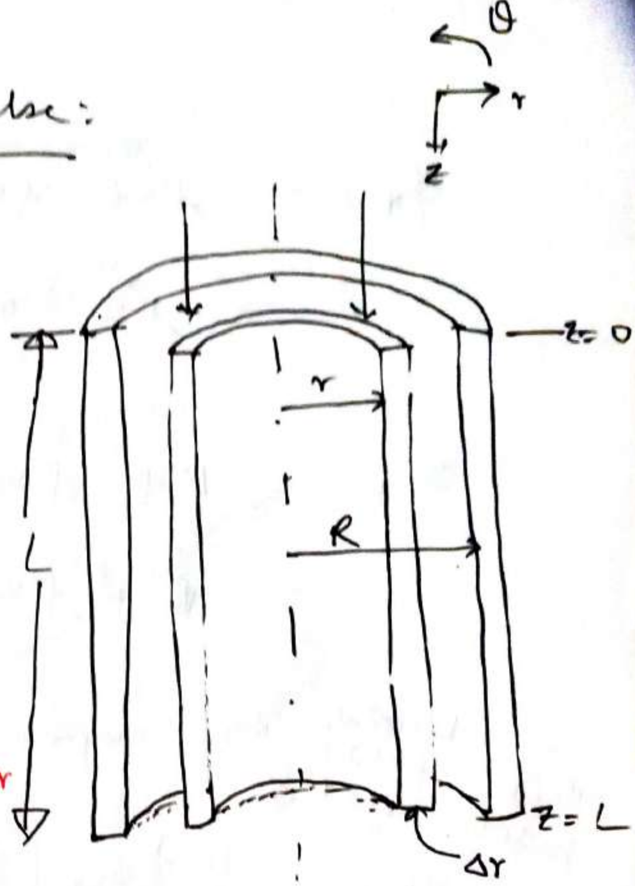
$$= \int_0^L \int_0^W \left(-\mu \frac{dv_z}{dx} \bigg|_{x=\delta} \right) dy \cdot dz$$

$$= LW \cdot (-\mu) \left(-\frac{\rho g \delta \cos\beta}{\mu} \right) = \rho g \delta L W \cos\beta$$

(z component of the weight of the film)

(44)

Flow Through a Circular Tube:



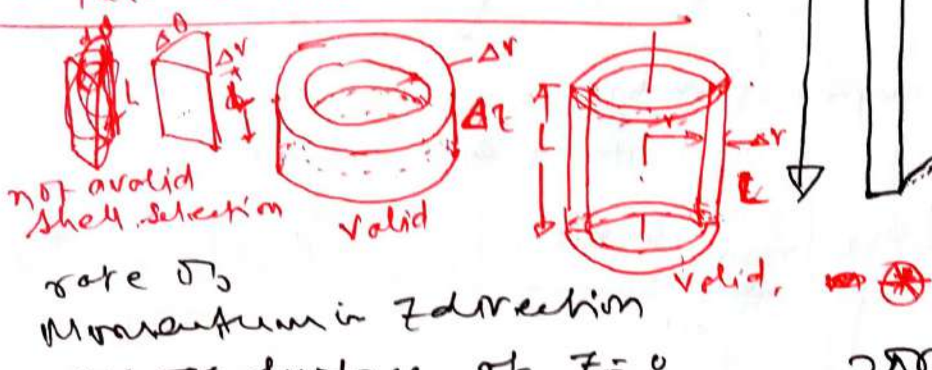
consider steady state flow

Notice

$v_z = v_z(r)$

v_r & $v_\theta = 0$

Also Talk about Shell Selection:



rate of Momentum in z direction across surface at $z = 0$ $\quad 2\pi r \cdot \Delta r \; \phi_{zz}\big|_{z=0}$.

- - - . Out
- - - - - . at $z = L$ $\quad 2\pi r \Delta r \; \phi_{zz}\big|_{z=L}$.

rate of moment in across surface at r $\quad 2\pi r L \; \phi_{rz}\big|_{r=r}$

rate of moment out across surface at $r = r + \Delta r$. $\quad 2\pi(r+\Delta r) \cdot L \; \phi_{rz}\big|_{r=r+\Delta r}$

Weight of fluid (gravity force) $= 2\pi r \Delta r \cdot L \cdot \rho g$.

Balance in z direction

$$\Rightarrow \quad 2\pi r \Delta r \; \phi_{zz}\big|_{z=L} - 2\pi r \cdot \Delta r \; \phi_{zz}\big|_{z=0} - \left(2\pi r L \; \phi_{rz}\big|_{r=r} - 2\pi(r+\Delta r) \cdot L \; \phi_{rz}\big|_{r+\Delta r}\right) = 2\pi r \Delta r \cdot L \; \rho g.$$

44a

Shear stress in cylindrical coordinate

$$\tau_{rr} = -\mu\left[2\frac{\partial v_r}{\partial r}\right] + \left(\frac{2}{3}\mu - k\right)(\nabla \cdot v)$$

$$\tau_{\theta\theta} = -\mu\left[2\frac{\partial v_\theta}{\partial \theta}\right] + \left(\frac{2}{3}\mu - k\right)(\nabla \cdot v)$$

$$\tau_{zz} = -\mu\left[2\frac{\partial v_z}{\partial z}\right] + \left(\frac{2}{3}\mu - k\right)(\nabla \cdot v)$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta z} = \tau_{z\theta} = -\mu\left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right]$$

$$\tau_{zr} = \tau_{rz} = -\mu\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]$$

Flow through a circular tube: Supplementary sheet

$v_r = 0$ | r momentum (neglected)

$v_\theta = 0$ | θ momentum (—"—)

$v_z =$ → z momentum

$v_z = f(r, \cancel{\theta}, \cancel{z})$

$\phi_{rz}, \quad \phi_{\theta z}, \quad \phi_{zz}$ will be of our concern.

$$\underline{\underline{\phi}} = p \cdot \underline{\underline{\delta}} + \underline{\underline{\tau}} + \rho \underline{v}\,\underline{v}$$

$$\phi_{rz} = p \cdot \cancel{0}^{\,0} + \tau_{rz} + \rho v_r \cancel{v_z}^{\,0} \qquad \left[\tau_{rz} = -\mu\left(\frac{\partial v_z}{\partial r} + \cancel{\frac{\partial v_r}{\partial z}}^{\,0}\right)\right]$$

$$\phi_{\theta z} = p \cdot \cancel{0}^{\,0} + \tau_{\theta z} + \rho v_\theta \cancel{v_z}^{\,0} \qquad \left[\tau_{\theta z} = -\mu\left[\frac{1}{r}\cancel{\frac{\partial v_z}{\partial \theta}}^{\,0} + \cancel{\frac{\partial v_\theta}{\partial z}}\right]\right]$$

$$\phi_{zz} = p \cdot 1 + \tau_{zz} + \rho v_z^2$$

$$= p + \left[-2\mu \cdot \cancel{\frac{\partial v_z}{\partial z}}^{\,0}\right] + \rho v_z^2$$

$$\phi_{rz} = \cancel{p}^{0} \; \rho v_r \cdot v_z + \tau_{rz}$$

$$= \cancel{p}^{0} \; 0 + \left(-\mu \frac{\partial v_z}{\partial r}\right)$$

$$\phi_{zz} = p + \rho v_z \cdot v_z + \tau_{zz}$$

$$= p + \rho v_z v_z + \left(-2\mu \cancel{\frac{\partial v_z}{\partial z}}^{0}\right)$$

→ will be same at both ends of the tube

Divide the expression by $2\pi \Delta r \cdot L$ and take lim $\Delta r \to 0$

This 'r' refers to position '$r+\Delta r$'

$$\lim_{\Delta r \to 0} \left(\frac{r\phi_{rz}\big|_{r+\Delta r} - r\phi_{rz}\big|_{r}}{\Delta r}\right) = \left(\frac{\phi_{zz}\big|_{z=0} - \phi_{zz}\big|_{z=L}}{L} + \rho g\right) r$$

This r cannot be cancelled out of it also refers to $(r+\Delta r)$ at one place.

$$\frac{\partial}{\partial r}(r\phi_{rz}) = \left(\frac{\phi_{zz}\big|_{z=0} - \phi_{zz}\big|_{z=L}}{L} + \rho g\right) \cdot r$$

$$\frac{\partial}{\partial r}(r\phi_{rz}) = \left(\{p_0 - \rho g \cdot (0)\} - \{p_L - \rho g \cdot L\} \cdot \frac{r}{L}\right)$$

$$= \frac{\mathcal{P}_0 - \mathcal{P}_L}{L} \cdot r$$

→ Modified pressure

$$\frac{\partial}{\partial r}(r\tau_{rz}) = \frac{\mathcal{P}_0 - \mathcal{P}_L}{L} \cdot r$$

integrating

$$r\tau_{rz} = \frac{\mathcal{P}_0 - \mathcal{P}_L}{L} \cdot \frac{r^2}{2} + C_1$$

$$\tau_{rz} = \frac{\mathcal{P}_0 - \mathcal{P}_L}{2L} \cdot r + \frac{C_1}{r}$$

<u>B.C.</u> at $r = 0$, τ_{rz} = finite thus $C_1 = 0$

Otherwise τ_{rz} will be infinity

$$\therefore \boxed{\tau_{rz} = \frac{P_0 - P_1}{2L} \cdot r}$$

Now

$$\tau_{rz} = -\mu \frac{\partial v_z}{\partial r} = \frac{P_0 - P_1}{2L} \cdot r$$

$$v_z = -\frac{P_0 - P_1}{2L\mu} \cdot \frac{r^2}{2} + c_2$$

at $r = R$, $v_z = 0$, (No slip condition)

v_{fluid} = velocity of fluid solid interface

$$0 = -\frac{P_0 - P_1}{2L\mu} \cdot \frac{R^2}{2} + c_2$$

$$\therefore c_2 = \frac{P_0 - P_1}{4\mu L} \cdot R^2$$

$$\therefore \boxed{v_z = \frac{P_0 - P_1}{4\mu L} R^2 \left[1 - \left(\frac{r}{R}\right)^2\right]}$$

Parabolic velocity distribution

(i) Max Velocity

$$V_{z\,max} = \frac{P_0 - P_1}{4\mu L} \cdot R^2$$

(ii) avg velocity

$$\langle v_z \rangle = \frac{\int_0^{2\pi} \int_0^R v_z \, r\,dr.\,d\theta}{\int_0^{2\pi} \int_0^R r\,dr.\,d\theta}$$

=

$$\langle v_z \rangle = \frac{(P_0 - P_L) \cdot R^2}{8 \mu L} = \frac{1}{2} \cdot v_{z\,max}$$

(iii) Mass flow rate

$$W = \rho \cdot \langle v_z \rangle \cdot \pi R^2$$

$$= \rho \cdot \frac{(P_0 - P_L)}{8 \mu L} \cdot R^2 \cdot \pi R^2 = \frac{\pi \rho (P_0 - P_L) R^4}{8 \mu L}$$

Hagen - Poiseuille eqn; It is used for measuring viscosity in a capillary viscometer
see Bird Ex. 2.3-1

(IV) The z component of the force F of the fluid on the wetted surface of the pipe is $- \tau_{rz} \times$ area

$$F_z = \left(-\mu \frac{\partial v_z}{\partial r}\right) \cdot (2\pi R L) = \pi R^2 (P_0 - P_L)$$

viscous force ← $F_z = \pi R^2 (P_0 - P_L) + \pi R^2 L \cdot \rho \cdot g$
(pressure) (gravity)

Viscous force is counter balanced by pressure and gravity force.

Valid assumption in this derivation

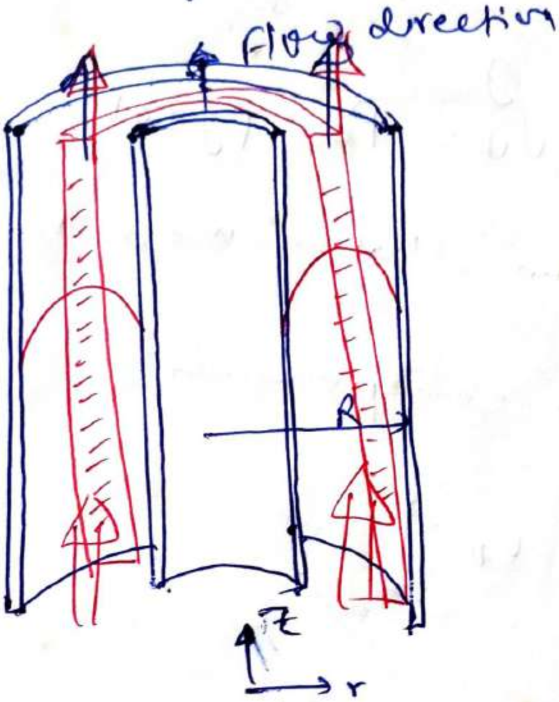
(i) laminar flow $Re < 2100$ (ii) constant - incompressible flow
(iii) S.S. flow (iv) Newtonian fluid. (v) End effects are neglected. $Le \geq 0.035D$
(vi) fluid is continuum (vii) No slip B.C.

Self read.
2.5 & 2.6

~~Both~~ No Example . 2.6-1

Assign. Mnt: 2A.1, 2A.3, 2A.4, 2B.6, 2B.7, 2B.12

Flow through annulus:



Can we use this eqn derived in the previous problem.

Ans: As the shell have same structure same eqn can be used, but the gravity term will here carry a -ive sign

eqn is

$$2\pi r.\Delta r \left[\left(\rho v_z^2 |_{z=0} - \rho v_z^2 |_{z=L} \right) + (P_0 - P_L) \right] + 2\pi L \cdot \left[r\tau_{rz}|_r - r\tau_{rz}|_{r+\Delta r} \right]$$

$$\ominus \rho g \, 2\pi r \Delta r . L = 0$$

↳ notice the negative sign

Therefore here

$$\frac{(P_0 - P_L)}{L} r + \left(\frac{r\tau_{rz}|_r - r\tau_{rz}|_{r+\Delta r}}{\Delta r} \right) - \rho g . r = 0$$

Modified pressure can be defined as

$$\mathcal{P}_L = P_L + \rho g L \quad ; \quad \mathcal{P}_0 = P_0 + \rho g (0)$$

how to make sure the eqn can be used

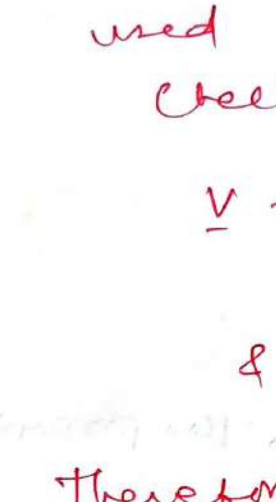
Check velocity components

$$\underline{V} = (\cancel{V_r}^{0}, \cancel{V_\theta}^{0}, V_z)$$

& $v_z = f(r)$

Therefore the shell solution will be same which will lead to same eqn

Q.



in this case which velocity component will exist

$\cancel{V_r}^{0}, \cancel{V_\theta}^{0}, V_z$

$V_z = f(r, \theta)$

$V_z \neq f(z)$

In this case we have

V_z & $V_r \rightarrow = f(r, z)$

$\neq f(\theta)$

$V_z = f(r, z)$

(47c)

$$-\frac{\partial r\cdot\tau_{rz}}{\partial r} = -\left(\frac{P_0 - P_L}{L}\right)\cdot r$$

Now check for the boundary condition
here we can not use @ $r = 0$
because there is no fluid at $r=0$, (out of our consideration)

$$r\cdot\tau_{rz} = +\left(\frac{P_0 - P_L}{L}\right)\frac{r^2}{2} + C_1$$

$$\tau_{rz} = -\mu\frac{\partial v_z}{\partial r} = -\mu\frac{dv_z}{dr} \quad \text{as } v_z \neq f(\theta, z)$$

$$-r\mu\frac{\partial v_z}{\partial r} = \left(\frac{P_0 - P_L}{L}\right)\frac{r^2}{2} + C_1$$

we don't know where the velocity is maximum —

let at $r = \lambda R$, velocity is max and hence $\frac{\partial v_z}{\partial r} = 0$.

$$C_1 = -\left(\frac{P_0 - P_L}{L}\right)\frac{\lambda^2 R^2}{2}$$

$$\therefore \tau_{rz} = \left(\frac{P_0 - P_L}{2L}\right)\left[r - \frac{\lambda^2 R^2}{r}\right] = \left(\frac{P_0 - P_L}{2L}\right)R\left[\frac{r}{R} - \lambda^2\left(\frac{R}{r}\right)\right]$$

(4.7d)

Further

$$\frac{dv_z}{dr} = -\frac{(P_0 - P_L)R}{2\mu L}\left[\frac{r}{R} - \lambda^2\left(\frac{R}{r}\right)\right]$$

as we $\left(\frac{r}{R}\right)$ R being constant

$$\frac{d}{dr}(r/R) = \frac{1}{R}$$

$$\therefore dr = R\,[d(r/R)]$$

$$\therefore \frac{dv_z}{R\,d\left(\frac{r}{R}\right)} = -\left(\frac{P_0 - P_L}{2\mu L}\right)R\left[\frac{r}{R} - \lambda^2\left(\frac{R}{r}\right)\right]$$

or

$$dv_z = -\frac{(P_0 - P_L)}{2\mu L}R^2\left[\frac{r}{R} - \lambda^2\left(\frac{R}{r}\right)\right]d(r/R)$$

Integrate

$$v_z = -\left(\frac{P_0 - P_L}{2\mu L}\right)R^2\left[\left(\frac{r}{R}\right)^2\cdot\frac{1}{2} - \lambda^2\ln\left(\frac{r}{R}\right)\right] + C_2'$$

$$= -\left(\frac{P_0 - P_L}{4\mu L}\right)R^2\left[\left(\frac{r}{R}\right)^2 - 2\lambda^2\ln\left(\frac{r}{R}\right)\right] + C_2'$$

$$= -\left(\frac{P_0 - P_L}{4\mu L}\right)R^2\left[\left(\frac{r}{R}\right)^2 - 2\lambda^2\ln\left(\frac{r}{R}\right) + C_2\right]$$

B.C. @ $r = \kappa R$ $\quad v_z = 0,$

@ $r = R$ $\quad v_z = 0.$

Two B.C's because we have two unknowns λ, & C_2 in the eqn

From 1st B.C

$$0 = -A\left[\kappa^2 - 2\lambda^2 \ln \kappa + C_2\right]$$

From IInd B.C

$$0 = -A\left[1 + C_2\right] \qquad \text{where } A = \frac{(P_0 - P_L)R^2}{4\mu L}$$

$$\therefore C_2 = -1$$

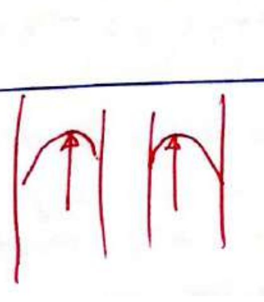
& from 1st eqn

$$\boxed{2\lambda^2 = \frac{\kappa^2 - 1}{\ln \kappa}} = \frac{1-\kappa^2}{\ln(1/\kappa)}$$

Now

$$\tau_{rz} = \left(\frac{P_0 - P_L}{2L}\right)R\left[\left(\frac{r}{R}\right) - \frac{1-\kappa^2}{2\ln(1/\kappa)}\left(\frac{R}{r}\right)\right]$$

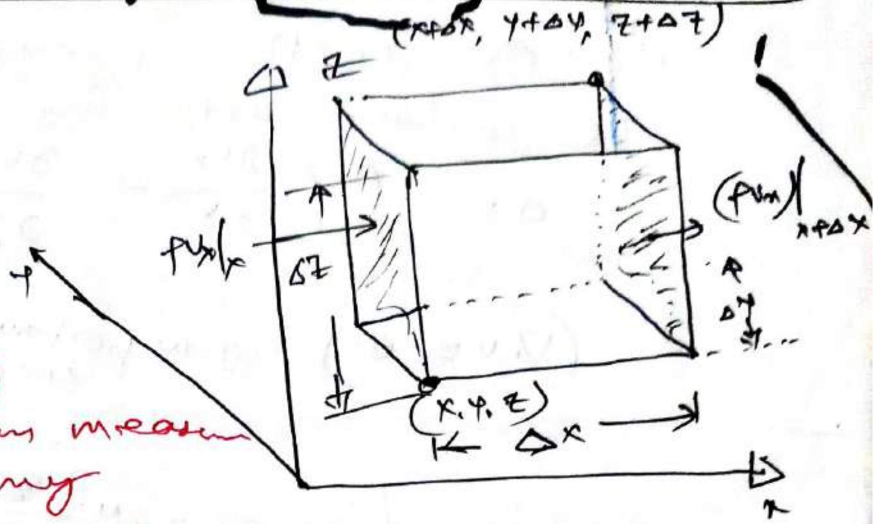
$$\boxed{v_z = \left(\frac{P_0 - P_L}{4\mu L}\right)R^2\left[1 - \left(\frac{r}{R}\right)^2 - \frac{1-\kappa^2}{\ln(1/\kappa)}\ln\left(\frac{R}{r}\right)\right]}$$



Max velocity will be close to inner pipe.

The Equations of change for Isothermal System.

Refer to the lecture videos of Prof. Sunando.
for further explanation, Lecture 9 to 16.



we call it eqn of change as we can measure the change of any quantity using this eqn.

The eqn of continuity: (Change in mass)

Mass balance over the element (Isothermal System)

$$\left\{\begin{array}{c}\text{rate of}\\ \text{increase}\\ \text{of mass}\end{array}\right\} = \left\{\begin{array}{c}\text{rate of}\\ \text{mass in}\end{array}\right\} - \left\{\begin{array}{c}\text{rate of}\\ \text{mass}\\ \text{out}\end{array}\right\}$$

$$\rho v_x|_x \cdot \Delta y \cdot \Delta z - \rho v_x|_{x+\Delta x}\,\Delta y \cdot \Delta z = \frac{\partial \rho}{\partial t}\cdot \Delta x\,\Delta y\,\Delta z$$

→ Area ⊥ to flow

$$+ \left(\rho v_y|_y - \rho v_y|_{y+\Delta y}\right)\Delta z \cdot \Delta x$$

$$+ \left(\rho v_z|_z - \rho v_z|_{z+\Delta z}\right)\Delta x \cdot \Delta y$$

divide by Δx, Δy, Δz and take limit $\Delta x, \Delta y, \Delta z \to 0$

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right)$$

$$\boxed{\frac{\partial \rho}{\partial t} = -(\nabla \cdot \rho v)}$$ [eqn of continuity]

(∇ → vector; ρv → vector; $\frac{\partial \rho}{\partial t}$ → scalar)

L.H.S. → net rate of mass increase / volume

R.H.S. → net rate of mass addition due to convective flow.

$\nabla \cdot \rho v \rightarrow$ divergence of ρv.

if $\rho =$ const. (incompressible fluid)
i.e. no change w.r.t. time and position

$$0 = \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$

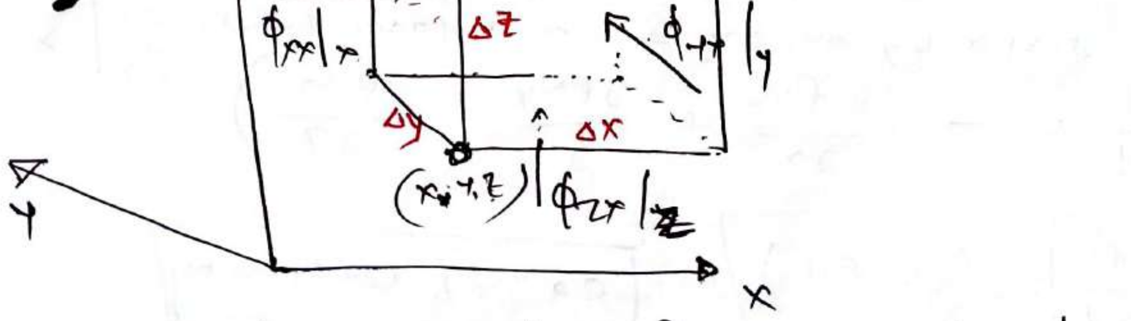
$(\nabla \cdot v = 0)$ Important conclusion

$$\tau_{zz} \Big|_{z=0} = -2\mu \frac{\partial v_z}{\partial z} \Big|_{z=0} = 2\mu \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) \Big|_{z=0} = 0$$

no slip condition $\frac{\partial v_x}{\partial x}, \frac{\partial v_y}{\partial y} = 0$

The equation of Motion:

$$\begin{bmatrix} \text{rate of} \\ \text{increase of} \\ \text{momentum} \end{bmatrix} = \begin{bmatrix} \text{rate of} \\ \text{momentum} \\ \text{in} \end{bmatrix} - \begin{bmatrix} \text{rate of} \\ \text{momentum} \\ \text{out} \end{bmatrix} + \begin{bmatrix} \text{external force on} \\ \text{the fluid} \end{bmatrix}$$



$$\Delta y \cdot \Delta z \left(\phi_{xx}|_x - \phi_{xx}|_{x+\Delta x} \right) + \Delta z \cdot \Delta x \left(\phi_{yx}|_y - \phi_{yx}|_{y+\Delta y} \right) + \Delta x \cdot \Delta y \left(\phi_{zx}|_z - \phi_{zx}|_{z+\Delta z} \right)$$

which is the net rate of x-momentum

External force (typically a gravitation force)

$\rho g_x \Delta x \Delta y \Delta z$

rate of increase of x-momentum within the volume element: $\Delta x . \Delta y \Delta z \left(\frac{\partial (\rho v_x)}{\partial t} \right)$

(★) Adding above terms and dividing by $\Delta x . \Delta y . \Delta z$

$$\frac{\partial}{\partial t}(\rho v_x) = -\left(\frac{\partial}{\partial x}\phi_{xx} + \frac{\partial}{\partial y}\phi_{yx} + \frac{\partial}{\partial z}\phi_{zx}\right) + \rho g_x$$

likewise for y & z direction

$$\frac{\partial}{\partial t}(\rho v_y) = -\left(\frac{\partial}{\partial x}\phi_{xy} + \frac{\partial}{\partial y}\phi_{yy} + \frac{\partial}{\partial z}\phi_{zy}\right) + \rho g_y$$

&

$$\frac{\partial}{\partial t}(\rho v_z) = -\left(\frac{\partial}{\partial x}\phi_{xz} + \frac{\partial}{\partial y}\phi_{yz} + \frac{\partial}{\partial z}\phi_{zz}\right) + \rho g_z$$

Using vector tensor notation

$$\partial/\partial t(\rho v_i) = -[\nabla . \phi]_i + \rho g_i \qquad i = x, y, z$$

notice the square brackets used for vector tensor notation

~~then the ith component of~~

Quantities ρv_i are cartesian components of the vector ρv which is the momentum per unit volume at a point in the fluid. Likewise ρg_i are the components of ρg vector. The $[\nabla . \phi]_i$ is the ith component of $[\Delta . \phi]$ vector.

in general

$$\frac{\partial}{\partial t}(\rho v) = -[\nabla . \phi] + \rho g$$

$$\phi = p\delta + \rho v v + \tau \quad \text{combined momentum flux}$$

$$\frac{\partial}{\partial t}(\rho v) = -[\nabla . \rho v v] - \nabla p - [\nabla . \tau] + \rho g$$

(scalar: ∇p)

(★) → rate of momentum (in - out) + sum of forces = rate of acc. of momentum

(51)

Physical meaning of the term:

(1) term $\longrightarrow \frac{\partial}{\partial t} \rho v \rightarrow$ rate of increase of momentum / volume

(2) term rate of momentum addition by convection / volume

(3) + (4) $\longrightarrow$ due to molecular transport / volume

(5) external force on fluid / volume.

$\nabla P \longrightarrow$ "grad P"

~~The Equation of Mechanical Energy:~~

The Equation of ~~Mechanical Energy~~: Change in terms of the substantial derivative.

Partial derivative :- $c = f(x, y, z, t)$

$$\left.\frac{\partial c}{\partial t}\right|_{x,y,z} \rightarrow \text{standing at one point}$$

Total time ~~Substantial~~ derivative. $\left(\frac{d}{dt}\right)$

$$\frac{dc}{dt} = \left.\frac{\partial c}{\partial t}\right|_{x,y,z} + \frac{dx}{dt}\left.\left(\frac{\partial c}{\partial x}\right)\right|_{y,z,t} + \frac{dy}{dt}\left.\left(\frac{\partial c}{\partial y}\right)\right|_{x,z,t} + \frac{dz}{dt}\left.\left(\frac{\partial c}{\partial z}\right)\right|_{x,y,t}$$

$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rightarrow$ velocity of the boat in x, y, & z direction

Substantial time Derivative:

$$\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + v_x\frac{\partial c}{\partial x} + v_y\frac{\partial c}{\partial y} + v_z\frac{\partial c}{\partial z}$$

$$= \frac{\partial c}{\partial t} + (v \cdot \nabla c)$$

$$\boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + (v \cdot \nabla)}$$ Substantial derivative

time rate of change as one moves with the substance. Also called derivative following the motion.

converting expression in terms of $\frac{\partial}{\partial t}$ into $\frac{D}{Dt}$

for any scalar function $f(x, y, z, t)$

say we have

$$\frac{\partial}{\partial t}(\rho f) + \frac{\partial}{\partial x}(\rho v_x f) + \frac{\partial}{\partial y}(\rho v_y f) + \frac{\partial}{\partial z}(\rho v_z f)$$

$$= \rho\left(\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + v_z \frac{\partial f}{\partial z}\right) + f\left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho v_x + \frac{\partial}{\partial y}\rho v_y + \frac{\partial}{\partial z}\rho v_z\right)^{0}$$

due to eqn of continuity

$$= \rho \frac{Df}{Dt}$$

in vector form

$$\frac{\partial}{\partial t}(\rho f) + (\nabla \cdot \rho \underline{v} f) = \rho \frac{Df}{Dt}$$

if $f(x, y, z, t)$ is a vector function.

$$\frac{\partial}{\partial t}(\rho \underline{f}) + [\underline{\nabla} \cdot \rho \underline{v} \underline{f}] = \rho \frac{D\underline{f}}{Dt}$$ [v, f are vectors]

See Table 3.5-1 for various $\frac{D}{Dt}$ forms of eqn of change

~~SIMPLIFICATIONS OF~~

3.1-4 $\frac{D\rho}{Dt} = -\rho(\nabla \cdot \vec{v})$ (A) Equation of continuity in substantial derivative form

3.2-9 $\rho \frac{Dv}{Dt} = -\nabla p - [\nabla \cdot \tau] + \rho \vec{g}$ Equation of motion in susb. derv. form

3.3-1 $\rho \frac{D}{Dt}(\frac{1}{2} v^2) = -(\vec{v} \cdot \nabla p) - (v \cdot [\nabla \cdot \tau]) + \rho(v \cdot g)$.

Energy equation in the substatial derivative form. This equation has been obtained by taking dot product of velocity v with the equation of motion.

Each term of the energy equation has the unit. rate of change of energy per unit volume.

(54)

The most common simplifications of the equation of motion.

(i) constant ρ and μ → fluids are incompressible; Newton's law is applicable for viscosity.

$$\rho \frac{D}{Dt}\bar{V} = -\nabla p + \mu \nabla^2 \vec{V} + \rho \vec{g}$$

Recall.

$$\rho \frac{D\bar{V}}{Dt} = -\nabla p - [\nabla \cdot \tau] + \rho g$$

eqn of motion

or

$$\rho \frac{D}{Dt}\bar{V} = -\nabla P + \mu \nabla^2 \vec{V}$$

P → modified pressure

Navier-Stokes eqn

It can not be used for gases. (ρ may not be constant)
Useful for liquid only.

(ii) When acceleration terms in the Navier-Stokes equation are neglected, that is when

$$\rho \frac{Dv}{Dt} = 0 \longrightarrow$$

$$0 = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g}$$

Stokes flow equation
or
creeping flow equation.

∴ $\rho \cdot [\vec{v} \cdot \nabla v]$ can be discarded when the flow is extremely slow

(iii) When viscous forces are neglected i.e. $[\nabla \cdot \tau] = 0$ the eqn of motion becomes

$$\rho \frac{DV}{Dt} = -\nabla p + \rho g$$

no viscous forces

This is Euler equation for inviscid flow.

Use of equation of change to solve flow problems

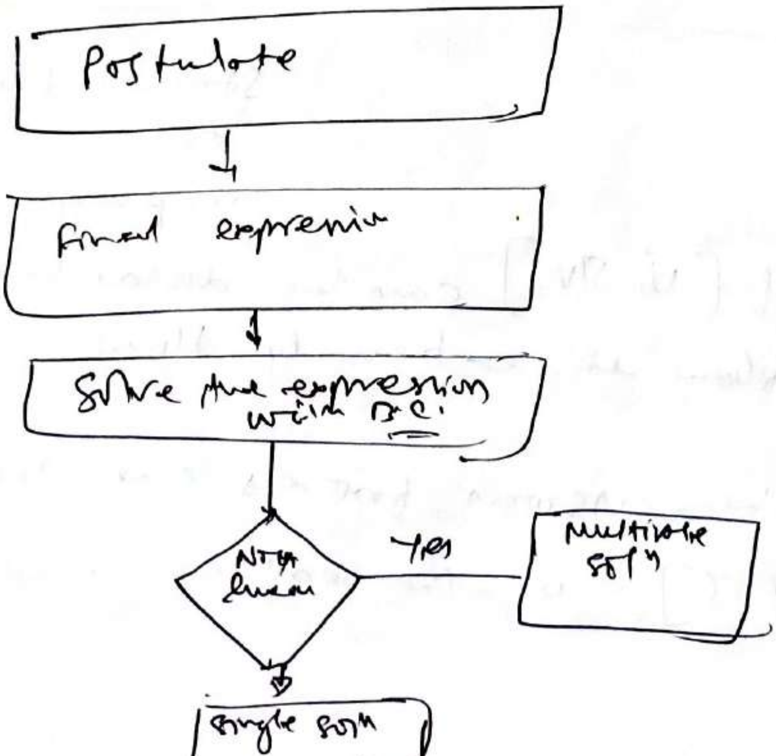
General Equations to be used

continuity, motion, eq^n of state $\rho = \rho(P)$, components of τ, viscosity eq^n $\mu = \mu(\rho)$

They are mostly used with some restrictions

(a) constant ρ constant μ

Additionally with some appropriate, B.C.

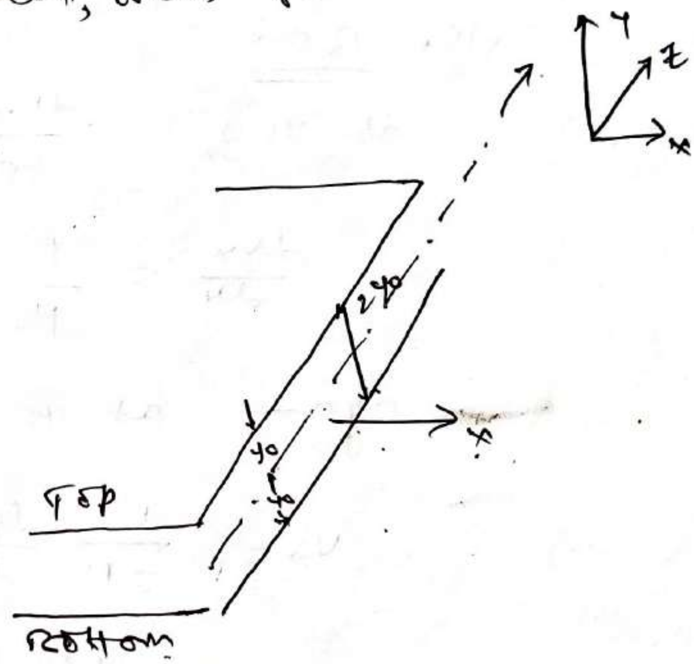


Ex. 3.8-1 — Geankoplis

Laminar flow b/w Horizontal Parallel plates

Steady state, constant density, constant μ fluid (liquid) flowing b/w two parallel plates of width.

Velocity profile at somewhere midway b/w inlet & outlet, with flow driven by pressure gradient



$v_x = v_x(y)$, $v_y = v_z = 0$, $\frac{\partial v_x}{\partial t} = 0$ (s.s.)

$\frac{\partial v_x}{\partial x}$, $\frac{\partial^2 v_x}{\partial x^2} = 0$,

from continuity eqn for constant density

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}^{\to 0} + \frac{\partial v_z}{\partial z}^{\to 0} = 0$$

The N-S. eqn of motion for x component

$$\rho\left(\frac{\partial v_x}{\partial t}^{\to 0} + v_x \frac{\partial v_x}{\partial x}^{\to 0} + v_y \frac{\partial v_y}{\partial y}^{\to 0} + v_z \frac{\partial v_z}{\partial z}^{\to 0}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2}^{\to 0} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}^{\to 0}\right) + \rho g_x$$

$$0 = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial y^2}\right) + \rho g_x \rightarrow 0 \qquad g_x = 0 \text{ (gravity only in vertical direction}$$

$$\frac{\partial P}{\partial x} = \mu \frac{\partial^2 v_x}{\partial y^2}$$

$$\frac{1}{\mu}\frac{dP}{dx} = \frac{d^2 v_x}{dy^2} = \text{const.}$$

$P \neq P(z)$
& if $2y_0 <<$ then $P \neq P(y)$
Now since $v_x \neq v_x(x)$
$\frac{dP}{dx} = \text{const.}$

Use B.C.:

at $y = 0$, $\frac{dv_x}{dy} = 0$, (from Symmetry)

$$\therefore \frac{dv_x}{dy} = \frac{1}{\mu}\frac{dP}{dx} \cdot y$$

again at $y = y_0$, $v_x = 0$,

$$\therefore v_x = \frac{1}{2\mu}\frac{dP}{dx} \cdot \left(y^2 - y_0^2\right)$$

$$v_x = \frac{1}{2\mu}\frac{dP\,y_0^2}{dx}\left[1 - \left(\frac{y_0}{y}\right)^2\right]$$

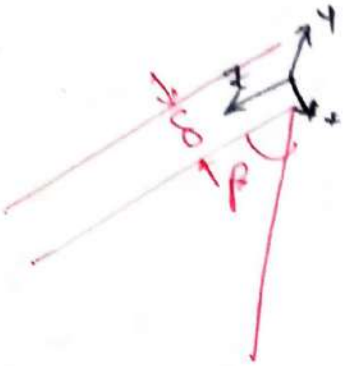
$v_x = v_{x\,max}$ at $y = 0$

$$v_{x,max} = \frac{1}{2\mu} \cdot \frac{dP}{dx}\left(-y_0^2\right)$$

or

$$v_x = v_{x\,max}\left[1 - \left(\frac{y}{y_0}\right)^2\right]$$

Flow of a falling film revisited: (ρ, μ constant)



Start with continuity eqn

$$\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho v) = 0$$

$$\nabla \cdot v = 0$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

$$v_x = v_y = 0$$

$$\therefore \frac{\partial v_z}{\partial z} = 0$$

hence

$$v_z \neq f(z)$$

N-S eqn can be applied as ρ & μ are constant

$$\rho \frac{Dv}{Dt} = -\nabla p + \mu \nabla^2 \bar{v} + \rho \bar{g}$$

x-comp

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x$$

$\therefore v_x = 0$; keep this ($\frac{\partial p}{\partial x}$)

y-comp

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y$$

$\therefore v_y = 0$

z-comp

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

(0: s.s.)

x – comp

$$0 = -\frac{\partial P}{\partial x} + \rho g \sin\beta$$

$$0 = -\frac{\partial p}{\partial x} + \rho g \sin\beta$$ (gives pressure profile)

$$0 = -\frac{\partial p}{\partial y}$$ therefore $p \neq f(y)$

$$0 = -\frac{\partial p}{\partial z} + \mu \frac{\partial^2 v_z}{\partial x^2} + \rho g \cos\beta$$ (→ (III))

→ gives velocity profile

Notice the II order differential eqn with two variable which may be difficult to solve.

Figure out which term may be zero in the eqn (III)

it is $\frac{\partial p}{\partial z} = 0$ (one may think logically)

∴ II nd is must to get velocity profile

$$\therefore -\mu \frac{\partial^2 v_z}{\partial x^2} = \rho g \cos\beta$$

which is similar to the ~~earlier~~ eqn we have seen earlier and can be solved with the similar B.C.s

~~Navier Stokes~~ eqn of motion in cylindrical co-ordinate system (Refer to appendix B5)

[SKIP to continuity eqn] → Appendix B.4

r- component

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\cdot\frac{\partial v_r}{\partial \theta}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r}\right)$$
$$=-\frac{\partial p}{\partial r}-\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{1}{r}\frac{\partial}{\partial \theta}\cdot\tau_{\theta r}+\frac{\partial}{\partial z}\tau_{zr}-\frac{\tau_{\theta\theta}}{r}\right)$$
$$+\rho g_r$$

θ- comp.

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r}\right)$$
$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}-\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{r\theta}\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta}+\frac{\partial}{\partial z}\tau_{z\theta}+\frac{\tau_{\theta r}-\tau_{r\theta}}{r}\right]+\rho g_\theta$$

z- comp

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\cdot\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)$$
$$=-\frac{\partial p}{\partial z}-\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta z}+\frac{\partial}{\partial z}\tau_{zz}\right)+\rho g_z$$

Continuity eqn in cylindrical co-ordinate

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0$$

~~Now switch to the flow through horizontal circular tube problem~~

N-S equation with constant ρ & μ B.6

r component

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\cdot\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right)$$
$$= -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r$$

θ- component

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\cdot\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right)$$
$$= -\frac{1}{r}\cdot\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta$$

z- comp

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right)$$
$$= -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

Switch to flow through straight horizontal pipe problem.

(58)

3.8.2 Assign Laminar flow b/w vertical Plates with one plate Moving

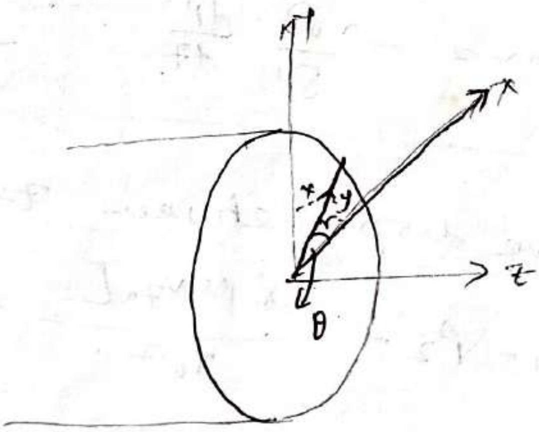
Differential Equations of continuity and Motion for flow in Stationary and Rotating cylinders.

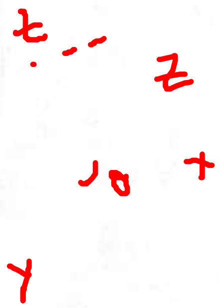
3.8.3 Laminar flow in a circular tube:

Derive the eqn for steady-state viscous flow in a horizontal tube of radius r_0, where the fluid is far from the inlet ~~and outlet~~ consider constant ρ & μ. One directional flow due to pressure gradient.

$V_x = V_y = 0$

~~$\frac{dv_z}{dz} = 0$ (from continuity~~





$z = z, \quad x = r\cos\theta, \quad y = r\sin\theta$

$\frac{dv_z}{dz} = 0$ from continuity eqn

from eqn of motion

$v_z = v_z(r)$

$$\frac{dp}{dz} = \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2}\right)$$

$r = +\sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{1}{\mu}\frac{dp}{dz} = \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 v_z}{\partial \theta^2}\right)$$

→ 0 as the flow is symmetric about z axis.

(59)

$$\frac{1}{\mu}\frac{dp}{dz} = \text{const} = \frac{d^2v_z}{dr^2} + \frac{1}{r}\frac{dv_z}{dr} = \frac{1}{r}\frac{d}{dr}\left(r \cdot \frac{dv_z}{dr}\right)$$

BC: at $r=0$, $\frac{dv_z}{dr}=0$ Symmetry condition

or $v_z=0$, at $r=r_0$ (tube radius)

$$\therefore v_z = \frac{1}{4\mu}\frac{dp}{dz}\left(r^2 - r_0^2\right)$$

$$v_z = v_{z,max}\left[1 - \left(\frac{r}{r_0}\right)^2\right]$$

$$v_{z,av} = -\frac{r_0^2}{8\mu}\frac{dp}{dz} \rightarrow \text{check}$$

Pressure drop between $z=0$, & $z=L$

$$P_1 - P_2 = \frac{8\mu v_{z,av} L}{r_0^2} = \frac{32\mu v_{z,av} L}{D^2}$$

Hagen-poiseuille eq^n

* Alternately one can directly start from cylindrical co-ordinate

$$\rho\left(\frac{\partial v_z}{\partial t}^{\nearrow 0} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\cdot\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z}$$

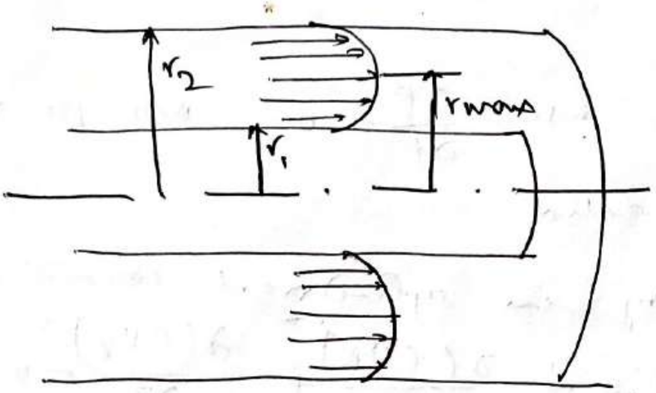
$$+ \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}^{\nearrow 0}\right] + \rho g_z$$

$$v_r = 0, \quad \frac{\partial v_z}{\partial \theta} = 0, \quad \frac{\partial v_z}{\partial z} = 0$$

3.8-4

Laminar flow in a cylindrical Annulus

Derive the equation for steady-state laminar flow inside the annulus b/w two concentric horizontal pipes



$\frac{dv_z}{dr} = 0$ at $r = r_{max}$ where the velocity is maximum

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{1}{\mu}\frac{dp}{dz}$$

$$r\frac{dv_z}{dr} = \frac{1}{\mu}\frac{dp}{dz}\left(\frac{r^2}{2} - \frac{r_{max}^2}{2}\right)$$

Use this B.C. for Ist integration

IInd B.C $v_z = 0$, at $r = r_1$

$$v_z = \frac{1}{2\mu}\frac{dp}{dz}\left(\frac{r^2}{2} - \frac{r_1^2}{2} - r_{max}^2 \ln\frac{r}{r_1}\right)$$

also

$v_z = 0$ at $r = r_2$

$$v_z = \frac{1}{2\mu}\frac{dp}{dz}\left(\frac{r^2}{2} - \frac{r_2^2}{2} - r_{max}^2 \ln\frac{r}{r_2}\right)$$

or

$$r_{max} = \sqrt{\frac{1}{\ln\left(\frac{r_2}{r_1}\right)}\frac{(r_2^2 - r_1^2)}{2}}$$

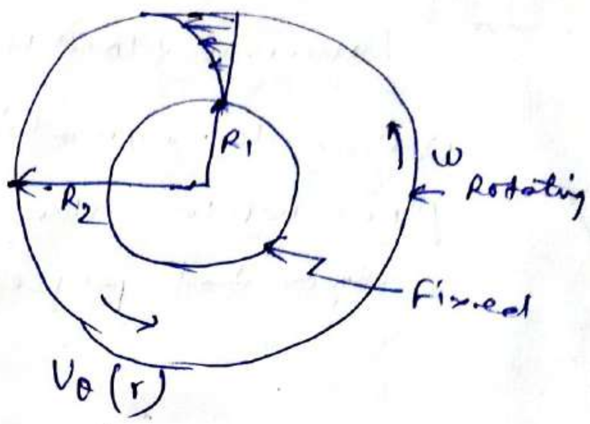
(61)

Ex. 3.8.5 Geankoplis — Assignment

ρ & μ are constant

Laminar flow.

Q. Find the shear stress and velocity distribution for the flow



$v_r = v_z = 0$, at s.s. $\frac{\partial \rho}{\partial t} = 0$ No pressure gradient in θ direction

∴ eqn of continuity in cylindrical co-ordinate

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

$$\rho \frac{\partial v_\theta}{\partial \theta} = 0$$

The eqn of motion in cylindrical coordinate

$$-\rho \frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r} \quad (r\text{- component})$$

$$0 = \frac{d}{dr}\left(\frac{1}{r}\frac{d(r v_\theta)}{dr}\right) \quad (\theta\text{- component})$$

$$0 = -\frac{\partial p}{\partial z} + \rho g_z \quad (z\text{- component})$$

from θ component

$$v_\theta = C_1 r + \frac{C_2}{r}$$

B.C. $r = R_1$ $v_\theta = 0$; at $r = R_2$; $v_\theta = \omega R_2$

$$C_1 R_1 + \frac{C_2}{R_1} = 0 \quad \Rightarrow C_2 = -C_1 R_1^2$$

$$\omega R_2 = C_1 R_2 + \frac{C_2}{R_2}$$

(62) $$\omega R_2 = c_1 R_2 + \frac{c_2 R_1^2}{R_2}$$

$$\Rightarrow \quad c_1 = \frac{\omega R_2^2}{(R_2^2 - R_1^2)} \; ; \quad c_2 = \frac{-\omega R_1^2 R_2^2}{(R_2^2 - R_1^2)}$$

$$\therefore V_\theta = c_1 r + \frac{c_2}{r} \Rightarrow \frac{\omega R_2^2 r}{(R_2^2 - R_1^2)} - \frac{\omega R_1^2 R_2^2}{(R_2^2 - R_1^2)} \cdot \frac{1}{r}$$

$$v_\theta = \frac{1}{r}\frac{\omega R_1^2 R_2^2}{(R_1^2 - R_2^2)} - \frac{\omega R_2^2 \cdot r}{(R_1^2 - R_2^2)}$$

$$= \frac{\omega}{(R_1^2 - R_2^2)} \left[\frac{R_1^2 R_2^2}{r} - \frac{R_2^2 \cdot r}{1} \right]$$

$$\boxed{v_\theta = \frac{\omega R_1 R_2^2}{(R_1^2 - R_2^2)} \left[\frac{R_1}{r} - \frac{r}{R_1} \right]}$$

SHEAR STRESS

$$\tau_{r\theta} = -\mu \left[r \frac{\partial (v_\theta / r)}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \quad (\text{second term} \to 0)$$

$$= -2\mu \omega R_2^2 \left(\frac{1}{r^2} \right) \left[\frac{R_1^2 / R_2^2}{1 - R_1^2 / R_2^2} \right]$$

Torque required to rotate the cylinder

$$T = (2\pi R_2 H)(-\tau_{r\theta})\Big|_{r=R_2} (R_2)$$

H → Height of the cylinder

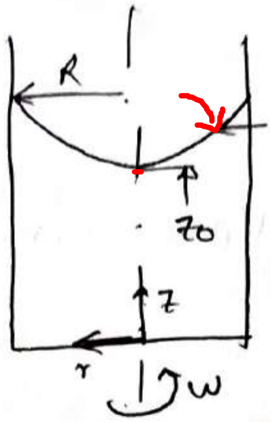
Assignment ex. 3.8-6 Geankoplis

Q. 3.F-6
~~geankoplis~~

Rotating liqd. in a cylindrical container

Q. At steady state find the shape of the free surface

$P = P(r, z)$



Sol^n

$v_r = v_z = 0, \quad g_r \;\&\; g_\theta = 0, \quad \& \; g_z = -g$

$$\rho \frac{v_\theta^2}{r} = \frac{\partial P}{\partial r} \quad (r\text{-component}) \quad -(1)$$

$$0 = \mu \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial (r v_\theta)}{\partial r}\right) \quad (\theta\text{-component}) \quad -(2)$$

$$\frac{\partial P}{\partial z} = -\rho g \quad (z\text{-component}) \quad (3)$$

from second eq^n

$$v_\theta = C_1 r + \frac{C_2}{r}$$

at $r = 0$, $v_\theta \to$ finite thus $C_2 = 0$

at $r = R$, $v_\theta = wR$

$\therefore v_\theta = C_1 r \quad \therefore C_1 = w$

$$\boxed{v_\theta = w \cdot r} \quad -(4)$$

from eq^n (1) & (4)

$$\therefore \frac{\partial P}{\partial r} = \rho w^2 r$$

also $\frac{\partial P}{\partial z} = -\rho g$

~~$P = P_0 = \frac{\rho w^2 r^2}{2}$~~

$$dP = \frac{\partial P}{\partial r} \cdot dr + \frac{\partial P}{\partial z} \cdot dz$$

$$= \rho w^2 r \cdot dr + (-\rho g) \cdot dz$$

$$P_0 = \frac{\rho w^2 r^2}{2} - \rho g z + C_3$$

(64) at $z = z_0$ $\quad p = p_0$ & $r = 0$,

$$\therefore p - p_0 = \frac{\rho \omega^2 r^2}{2} + \rho g (z_0 - z)$$

For free surface $\quad p = p_0$

$$\therefore \rho g (z - z_0) = \frac{\rho \omega^2 r^2}{2}$$

$$\boxed{(z - z_0) = \frac{\omega^2 r^2}{2g}}$$ Parabolic eqn.

Other Methods for solution of differential eqn of motion

In earlier discussion Navier-Stokes eqns were ~~solved~~ solved analytically where there was only one nonvanishing velocity component. For two or more than two non-vanishing components the problem becomes more complicated. This section ~~analyses~~ considers some approximation that simplify to the differential eqns to obtain ~~an~~ analytical soln.

Stream function:

$\psi(x, y)$ for two dimensional steady incompressible flow

such that $v_x = \frac{\partial \psi(x, y)}{\partial y}$

&

$$v_y = -\frac{\partial \psi}{\partial x}$$

This definition can be used to obtain a differential eqn for ψ that is equivalent to Navier-Stokes eqn

Physical significance of ψ: In steady flow lines streamlines are defined by ψ = constant which are actual curve traced by the particle of the fluid.

A stream function exists for all two dimensional steady, incompressible flow, viscous/inviscid rotational/irrotational

Example: Stream function and stream lines

Consider ψ given as $x \cdot y$. Find the components of velocity. Also plot the streamlines for a constant $\psi = 4$ & $\psi = 1$

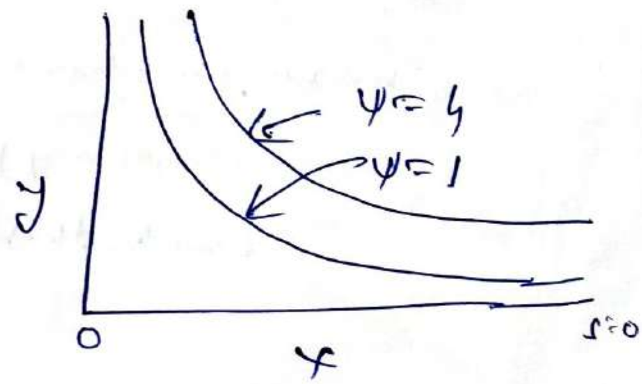
Now
$$V_x = \frac{\partial \psi}{\partial y} = x$$

$$V_y = -\frac{\partial \psi}{\partial x} = -y$$

Case I

$\psi = 1 = xy$

y	x
0.5	2
1.0	1
2.0	0.5
5.0	0.2
⋮	



→ irrotational, ρ = const. $\mu = 0$

(64a)

Potential flow and Velocity Potential (ϕ) → or potential fn.

define

$$v_x = \frac{\partial \phi(x,y)}{\partial x} \quad v_y = \frac{\partial \phi(x,y)}{\partial y}, \quad v_z = \frac{\partial \phi(x,y)}{\partial z}$$

This potential flow exist only for a flow with zero angular velocity, or irrotationality. This type of flow of an ideal / inviscid fluid (constant ρ & $\mu = 0$)

is called potential flow $\begin{cases} \rho \text{ constant} \\ \mu \to 0 \end{cases}$ inviscid, $\omega \to 0$ irrotational

Potential flow :- irrotational, nonviscous, constant density (incompressible)

Three dimensional flow

ϕ exist

ψ doesn't exist

Vorticity of a fluid:

$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 2\omega_z \quad \text{(Standard definition)}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2\omega_z$$

→ vorticity & ω_z in s^{-1} is angular velocity about the z axis.

If $2\omega_z = 0$ the flow is irrotational and a potential fn. exists.

Continuity $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$ or $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ | For constant ρ

Laplace's eqn

it can be solved with suitable for ϕ then v_x & v_y can be calculated.

(64b) likewise

for inviscid irrotational flow

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$ Laplace's eqn for stream fn.

equal Potential lines → lines of constant ϕ

and these line orthogonal to the lines of constant ψ everywhere for potential flow.

Proof.

define $d\psi = \frac{\partial \psi}{\partial x} . dx + \frac{\partial \psi}{\partial y} . dy$

for constant $\psi \rightarrow d\psi = 0$

$$\therefore \left(\frac{\partial \psi}{\partial x}\right) / \left(\frac{\partial \psi}{\partial y}\right) = - \frac{dy}{dx}$$

but $\frac{\partial \psi}{\partial x} = -v_y$, $\frac{\partial \psi}{\partial y} = v_x$

$$\therefore \left(\frac{dy}{dx}\right)_{\psi = const} = \left(\frac{v_y}{v_x}\right)$$

Also, for lines of constant ϕ

$d\phi = \frac{\partial \phi}{\partial x} . dx + \frac{\partial \phi}{\partial y} . dy = 0$

~~or $\left(\frac{dy}{dx}\right)_{\phi = const} = -\left(\frac{v_y}{v_x}\right) =$~~

$v_x . dx + v_y dy = 0$

$$\left(\frac{dy}{dx}\right)_{\phi = const} = - \frac{v_x}{v_y} = \frac{1}{-\left(\frac{dy}{dx}\right)_{\psi = const}}$$

(64c)

~~For a cylinder of infinite length~~

Consider a path in the xy plane such that ψ = constant. Along this path $d\psi = 0$.

White 10-2

$$\left(\frac{dy}{dx}\right)_{\psi = const} = \left(\frac{v_y}{v_x}\right)$$

The fn. ψ thus represents a stream function.

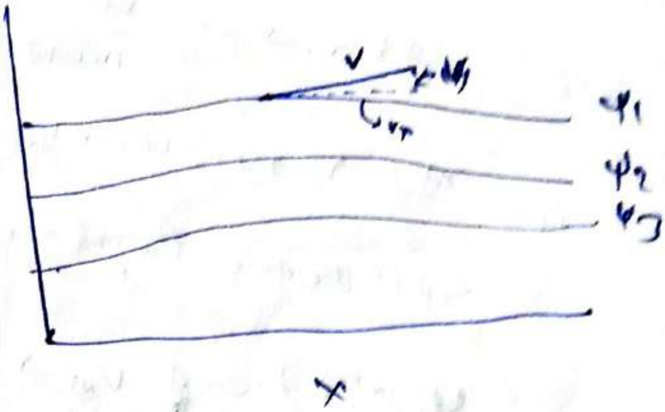
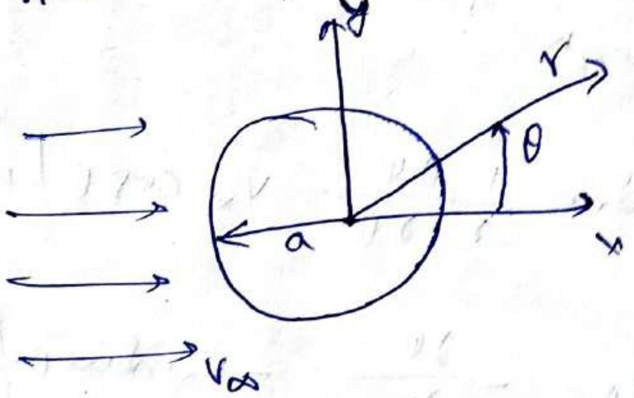


Fig. Stream line & stream fn.

Inviscid irrotational flow about an infinite length cylinder

White (10-3) Ex. of Use of stream fn. for a flow past a cylinder.



$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0$$ (Laplace eqn in cylindrical co-ordinates)

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad V_\theta = -\frac{\partial \psi}{\partial r}$$

Four B.C.s are required to solve it (2 B.C. in r, 2 B.C. in θ)

Now B.C.'s

1. The circle r = a must be a streamline. As there is zero velocity normal to a streamline

$$V_r\Big|_{r=a} = 0 \quad \text{or} \quad \frac{\partial \psi}{\partial \theta}\Big|_{r=a} = 0$$

(64d)

2. From symmetry the line θ = 0 must also be a streamline

$\therefore V_\theta \big|_{\theta=0} = 0$ or $\frac{\partial \psi}{\partial r}\Big|_{\theta=0} = 0$

3. As $r \to \infty$ velocity must be finite

4. At $r \to \infty$ $V \to V_\infty$ (a constant)

Soln to the laplacian eqn is

$$\psi(r, \theta) = V_\infty \cdot r \sin\theta \left[1 - \frac{a^2}{r^2}\right] \qquad r^2 = x^2 + y^2$$

The velocity components V_r & V_θ are obtained from eqn

$$V_r = \frac{1}{r} \cdot \frac{\partial \psi}{\partial \theta} = V_\infty \cos\theta \left[1 - \frac{a^2}{r^2}\right]$$

&

$$V_\theta = -\frac{\partial \psi}{\partial r} = -V_\infty \sin\theta \left[1 + \frac{a^2}{r^2}\right]$$

at $r = a$, velocity at the surface are

$V_r = 0$ &

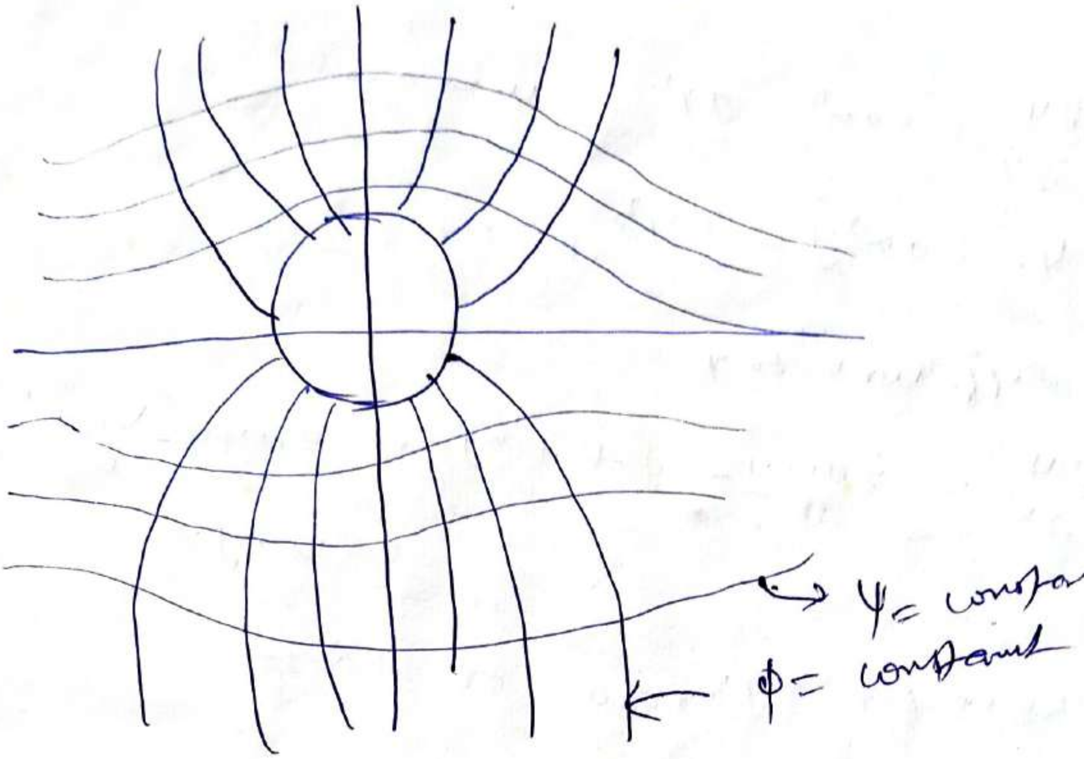
$V_\theta = -2V_\infty \sin\theta$

Notice no radial velocity as the cyl. surface is a streamline.

Now at $\theta = 0$, & $\theta = 180°$

$V_\theta = 0$, (stagnation point)

The inviscid, irrotational steady incompressible flow about an infinite cylinder



Ex. 9.2
Geankoplis

Stream function for a flow field / velocity comp for a flow field are

$v_x = a(x^2 - y^2)$ $\qquad$ $v_y = -2ayx$

Prove that it satisfied the conservation manner, determine ψ

Soln According to continuity eqn

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad (1)$$ in two dimensional flow

$$\frac{\partial v_x}{\partial x} = 2ax, \qquad \frac{\partial v_y}{\partial y} = -2ay$$

Hence the eq (1) is satisfied

Now

$v_x = \frac{\partial \psi}{\partial y} = ax^2 - ay^2$; $v_y = -\frac{\partial \psi}{\partial x} = -2ayx$

$\frac{\partial \psi}{\partial y} = ax^2 - ay^2$ integrate it

$\psi = ax^2y - \frac{ay^3}{3} + f(x)$

or diff. wrt to x

$\frac{\partial \psi}{\partial x} = 2axy - 0 + f'(x) = (-v_y)$

$= 2ayx$

therefor $f'(x) = 0$ or $f(x) = C$

$\therefore \psi = ax^2y - \frac{ay^3}{3} + C$

To plot ψ can be set equal to zero.

Stream fn. / potential fn. gives flow profile in the main body of the fluid and doesn't satisfy $v_x = v_y = 0$ on the wall surface.

Read sectn 3.10 of Geankoplis

Dimension less form of continuity eqn and eqn of motion

* We need dimension less form for scale-up of equipment.

Let define

$$\hat{x} = \frac{x}{l_0} \qquad l_0 \text{ is a characteristic length.}$$

$$\hat{y} = \frac{y}{l_0}$$

$$\hat{z} = \frac{z}{l_0}$$

$$\hat{v} = \frac{v}{v_0}$$

Check the ∇ operator

$$\nabla = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}$$

$$l_0 \nabla = \underline{i}\frac{\partial}{\partial\left(\frac{x}{l_0}\right)} + \underline{j}\frac{\partial}{\partial\left(\frac{y}{l_0}\right)} + k\frac{\partial}{\partial\left(\frac{z}{l_0}\right)}$$

$$\hat{\nabla} = \underline{i}\frac{\partial}{\partial \hat{x}} + j\frac{\partial}{\partial \hat{y}} + k\frac{\partial}{\partial \hat{z}}$$

or

$$\boxed{\hat{\nabla} = l_0 \nabla}$$

For constant ρ and μ eqn of continuity is

$$\nabla \cdot v = 0$$

$$\left(\frac{\hat{\nabla}}{l_0}\right) \cdot \left(\hat{v}\, v_0\right) = 0 \Rightarrow \frac{v_0}{l_0}\left(\hat{\nabla} \cdot \hat{v}\right) = 0$$

(62b) $(\hat{\nabla} \cdot \hat{v}) = 0$

Eqn of motion with constant ρ and μ

$$\rho \frac{Dv}{Dt} = -\nabla p + \mu \nabla^2 v + \rho g$$

$(p + \rho g)$

$$= -\nabla P + \mu \nabla^2 v$$

Modified pressure

force / vol. or Body force

$$\frac{D\hat{v}}{D\hat{t}} = -\hat{\nabla}\hat{P} + \left[\frac{\mu}{l_0 v_0 \rho}\right] \hat{\nabla}^2 \hat{v}$$

→ Reynolds number

$$\hat{t} = \frac{v_0 t}{l_0} \qquad \hat{P} = \frac{P - P_0}{\rho v_0^2}$$

$$\frac{D}{D\hat{t}} = \left(\frac{l_0}{v_0 t}\right) \frac{D}{Dt}$$

$$\nabla^2 = \frac{\partial^2}{\partial \hat{x}^2} + \frac{\partial^2}{\partial \hat{y}^2} + \frac{\partial^2}{\partial \hat{z}^2}$$

$$Fr = \left| \frac{v_0^2}{l_0 g} \right|$$ Froude Number

$$We = \left| \frac{\sigma}{l_0 v_0^2 \rho} \right|$$ → Weber Number

Syvum Info Page: https://www.syvum.com/cgi/online/serve.cgi/eng/fluid/fluid306.html

Transport Phenomena - Fluid Mechanics Problem :
Radial flow of a Newtonian fluid between parallel disks

Problem.

Steady, laminar flow occurs in the space between two fixed parallel, circular disks separated by a small gap $2b$. The fluid flows radially outward owing to a pressure difference $(P_1 - P_2)$ between the inner and outer radii r_1 and r_2, respectively. Neglect end effects and consider the region $r_1 \le r \le r_2$ only. Such a flow occurs when a lubricant flows in certain lubrication systems.

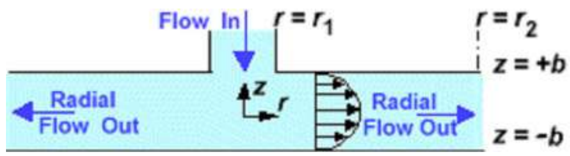


Figure. Radial flow between two parallel disks.

a) Simplify the equation of continuity to show that $r\, v_r = f$, where f is a function of only z.

b) Simplify the equation of motion for incompressible flow of a Newtonian fluid of viscosity μ and density ρ.

c) Obtain the velocity profile assuming creeping flow.

ketch the velocity profile $v_r\,(r, z)$ and the pressure profile $P(r)$.

e) Determine an expression for the mass flow rate by integrating the velocity profile.

f) Derive the mass flow rate expression in e) using an alternative short-cut method by adapting the plane narrow slit solution.

Solution.

Click here for stepwise solution

a)

Step. Simplification of continuity equation

Since the steady laminar flow is directed radially outward, only the radial velocity component v_r exists. The tangential and axial components of velocity are zero; so, $v_\theta = 0$ and $v_z = 0$.

For incompressible flow, the continuity equation gives $\nabla.\mathbf{v} = 0$.

In cylindrical coordinates,

$$\frac{1}{r}\frac{\partial}{\partial r}(r\, v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial r}(r\, v_r) = 0 \qquad (1)$$

On integrating the simplified continuity equation, $r\, v_r = f(\theta, z)$. Since the solution is expected to be symmetric about the z-axis, there is no dependence on the angle θ. Thus, f is a function of z only and not of r or θ. In other words, $r\,v_r = f(z)$. This is simply explained from the fact that mass (or volume, if density ρ is constant) is conserved; so, $\rho\,(2\,\pi\,r\,v_r\,dz) = dw$ is constant (at a given z) and is independent of r.

b)

Step. Simplification of equation of motion

For a Newtonian fluid, the Navier - Stokes equation is

$$\rho\,\frac{D\mathbf{v}}{Dt} = -\nabla P + \mu\nabla^2\mathbf{v} \tag{2}$$

in which P includes both the pressure and gravitational terms. On noting that $v_r = v_r(r, z)$, its components for steady flow in cylindrical coordinates may be simplified as given below.

r - component :

$$\rho\left(v_r\,\frac{\partial v_r}{\partial r}\right) = -\frac{\partial P}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\,v_r)\right) + \frac{\partial^2 v_r}{\partial z^2}\right] \tag{3}$$

θ - component :

$$0 = \frac{\partial P}{\partial \theta} \tag{4}$$

z - component :

$$0 = \frac{\partial P}{\partial z} \tag{5}$$

Recall that $r\,v_r = f(z)$ from the continuity equation. Substituting $v_r = f\,/\,r$ and $P = P(r)$ in equation (3) then gives

$$-\rho\,\frac{f^2}{r^3} = -\frac{dP}{dr} + \frac{\mu}{r}\frac{d^2 f}{dz^2} \tag{6}$$

c)

Step. Velocity profile

Equation (6) has no solution unless the nonlinear term (that is, the f^2 term on the left-hand side) is neglected. Under this 'creeping flow' assumption, equation (6) may be written as

$$r\,\frac{dP}{dr} = \mu\,\frac{d^2 f}{dz^2} \tag{7}$$

The left-hand side of equation (7) is a function of r only, whereas the right-hand side is a function of z only. This is only possible if each side equals a constant (say, C_0). Integration with respect to r from the inner radius r_1 to the outer radius r_2 then gives $P_2 - P_1 = C_0 \ln(r_2/r_1)$. On replacing C_0 in terms of f,

$$0 = (P_1 - P_2) + \left(\mu \ln \frac{r_2}{r_1} \right) \frac{d^2 f}{dz^2} \tag{8}$$

The above equation may be integrated twice with respect to z as follows.

$$\frac{df}{dz} = \frac{-\Delta P}{\mu \ln (r_2/r_1)} z + C_1 \tag{9}$$

$$f = \frac{-\Delta P}{2 \mu \ln (r_2/r_1)} z^2 + C_1 z + C_2 \quad \Rightarrow \quad v_r = \frac{-\Delta P}{2 \mu r \ln (r_2/r_1)} z^2 + C_1 \frac{z}{r} + \frac{C_2}{r} \tag{10}$$

Here, $\Delta P \equiv P_1 - P_2$. Equation (10) is valid in the region $r_1 \le r \le r_2$ and $-b \le z \le b$.

Imposing the no-slip boundary conditions at the two stationary disk surfaces ($v_z = 0$ at $z = \pm b$ and any r) gives $C_1 = 0$ and $C_2 = \Delta P\, b^2 / [2\, \mu \ln (r_2/r_1)]$. On substituting the integration constants in equation (10), the velocity profile is ultimately obtained as

$$v_r = \frac{\Delta P\, b^2}{2\, \mu\, r \ln (r_2/r_1)} \left[1 - \left(\frac{z}{b} \right)^2 \right] \tag{11}$$

d)

Step. Sketch of velocity profile and pressure profile

The velocity profile from equation (11) is observed to be parabolic for each value of r with $v_{r,max} = \Delta P\, b^2 / [2\, \mu\, r \ln (r_2/r_1)]$. The maximum velocity at $z = 0$ is thus inversely proportional to r. In general, it is observed from equation (11) that v_r itself is inversely proportional to r. Sketches of $v_r(z)$ for different values of r and $v_r(r)$ for different values of $|z|$ may be plotted.

The pressure profile obtained by integrating the left-hand side of equation (7) is $(P - P_2) / (P_1 - P_2) = [\ln(r/r_2)] / [\ln(r_1/r_2)]$. A sketch of $P(r)$ may be plotted which holds for all z.

e)

Step. Mass flow rate by integrating velocity profile

The mass flow rate w is rigorously obtained by integrating the velocity profile using $w = \int \mathbf{n} \cdot \rho \mathbf{v}\, dS$, where $\mathbf{n}$ is the unit normal to the element of surface area dS and $\mathbf{v}$ is the fluid velocity vector. For the radial flow between parallel disks, $\mathbf{n} = \boldsymbol{\delta}_r$, $\mathbf{v} = v_r \boldsymbol{\delta}_r$, and $dS = 2\pi r\, dz$. Then, substituting the velocity profile from equation (11) and integrating gives

$$w = \int_{-b}^{b} \rho\, v_r\, (2\, \pi\, r)\, dz = \frac{\pi\, \Delta P\, b^2 \rho}{\mu \ln (r_2/r_1)} \left[z - \frac{z^3}{3b^2} \right]_{-b}^{b} = \frac{4\pi\, \Delta P\, b^3 \rho}{3\mu \ln (r_2/r_1)} \tag{12}$$

f)

Step. Mass flow rate using short-cut method by adapting narrow slit solution

The plane narrow slit solution may be applied locally by recognizing that at all points between the disks the flow resembles the flow between parallel plates provided v_r is small (that is, the creeping flow is valid).

The mass flow rate for a Newtonian fluid in a plane narrow slit of width W, length L and thickness $2B$ is given by $w = 2\,\Delta P\, B^3\, W \rho / (3\, \mu\, L)$ (click here for derivation). In this expression, $(\Delta P/L)$ is replaced by $(-\,dP/dr)$, B by b, and W by $2\pi r$. Note that mass is conserved; so, w is constant. Then, integrating from r_1 to r_2 gives the same mass flow rate expression [equation (12)] as shown below.

$$w \int_{r_1}^{r_2} \frac{dr}{r} = \frac{4\pi\, b^3 \rho}{3\mu} \int_{P_1}^{P_2} (-dP) \qquad \Rightarrow \qquad w = \frac{4\pi\,(P_1 - P_2)\, b^3 \rho}{3\mu \ln(r_2/r_1)} \tag{13}$$

This alternative short-cut method for determining the mass flow rate starting from the narrow slit solution is very powerful because the approach may be used for non-Newtonian fluids where analytical solutions are difficult to obtain.

Related Problems in Transport Phenomena - Fluid Mechanics :

Transport Phenomena - Fluid Mechanics Problem : Newtonian fluid flow in a parallel - disk viscometer
ermination of the tangential velocity profile rather than the radial velocity profile for flow between two parallel, circular disks

UNIT-III

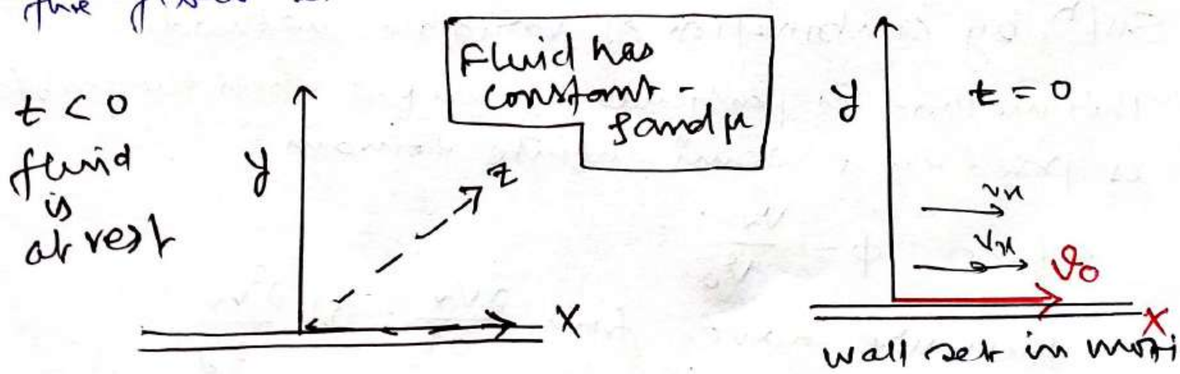
Unsteady state momentum transport:

Our interest may be in the region of transition where the change is w.r.t. time and space.

recall $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$ or $\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2}$

Flow near a wall suddenly set in motion:

Consider a semiinfinite body of liquid, with constant ρ and μ placed above a horizontal surface. Initially they are at rest and t=0 the solid surface is set in motion. Find the velocity v_x as function of y and t. There is no pressure gradient or gravity force in x direction. Consider the flow is laminar



Notice $v_x = v_x(y,t)$, $v_y = 0$, $v_z = 0$ (Unidirectional flow)

Let's apply eqn of motion

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial x} - [\nabla \cdot \tau] + \rho g_x \quad (g_x = 0)$$

$$= \mu \frac{\partial^2 v_x}{\partial y^2}$$

(66)

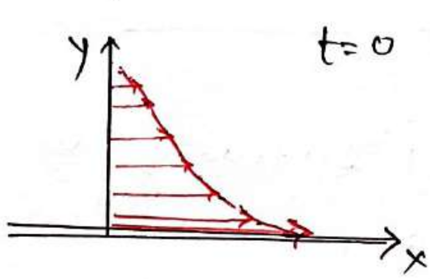
$$\rho \frac{\partial v_x}{\partial t} = \mu \frac{\partial^2 v_x}{\partial y^2}$$

$$\Rightarrow \frac{\partial v_x}{\partial t} = \frac{\mu}{\rho}\frac{\partial^2 v_x}{\partial y^2} \Rightarrow \boxed{\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}}$$

$\nu \rightarrow$ Kinematic viscosity

What Boundary conditions do we have?

$$v_x = v_x(y, t)$$



Initial condition

@ $t \leq 0$ $\quad v_x = 0$ $\quad$ for all y

B.C. 1 $\quad$ at $y = 0$, $\quad v_x = v_0 \;\; \forall \; t > 0$

B.C. 2 $\quad$ at $y = \infty$ $\quad v_x = 0 \;\; \forall \; t > 0$

Sol^n by combination of variable method.
This method is particularly useful when the problem is posed in a semi infinite domain.

define $\phi = \dfrac{v_x}{v_0}$

thus we have from $\dfrac{\partial v_x}{\partial t} = \nu \dfrac{\partial^2 v_x}{\partial y^2}$

$$\frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial y^2}$$

This is a PDE. and needs to be converted into an ODE. to do this let's rewrite the B.C.s & I.C.

$$\phi(y, 0) = 0, \quad \phi(0, t) = 1, \quad \phi(\infty, t) = 0$$

$\phi = \phi(\eta)$ $\eta \rightarrow$ dimensionless
as $\phi \rightarrow$ dimensionless.

$$\eta = \frac{y}{\sqrt{4\nu t}}$$

Came out of mathematical analysis.

$$\frac{\partial \phi}{\partial t} = \frac{d\phi}{d\eta} \cdot \frac{\partial \eta}{\partial t}$$ {From chain rule}

now

$$\frac{\partial \eta}{\partial t} = -\frac{1}{2}\frac{\eta}{t}$$

thus

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2}\frac{\eta}{t} \cdot \frac{d\phi}{d\eta}$$

likewise

$$\frac{\partial \phi}{\partial y} = \frac{d\phi}{d\eta} \cdot \frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{4\nu t}} \cdot \frac{d\phi}{d\eta}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{d\phi}{d\eta} \cdot \frac{1}{\sqrt{4\nu t}}\right) = \frac{1}{\sqrt{4\nu t}} \frac{\partial}{\partial y}\left(\frac{d\phi}{d\eta}\right) + \frac{d\phi}{d\eta} \cdot \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{4\nu t}}\right)^{\nearrow 0}$$

$$= \frac{1}{\sqrt{4\nu t}} \frac{d}{d\eta}\left(\frac{d\phi}{d\eta}\right) \cdot \frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{4\nu t}} \frac{d^2\phi}{d\eta^2} \cdot \frac{\partial \eta}{\partial y}$$

$$= \frac{1}{\sqrt{4\nu t}} \cdot \frac{d^2\phi}{d\eta^2} \cdot \frac{1}{\sqrt{4\nu t}} = \frac{1}{4\nu t} \cdot \frac{d^2\phi}{d\eta^2}$$

Thus

$$-\frac{1}{2}\frac{\eta}{t}\frac{d\phi}{d\eta} = \nu \cdot \frac{1}{4\nu t} \cdot \frac{d^2\phi}{d\eta^2}$$

or

$$\boxed{\frac{d^2\phi}{d\eta^2} + 2\eta \frac{d\phi}{d\eta} = 0}$$ An O.D.E.

B.C's can now be written as

@ $\eta = 0 \qquad \phi = 1$

$\eta = \infty \qquad \phi = 0$

Now let $\frac{d\phi}{d\eta} = \psi$

thus

$$\frac{d\psi}{d\eta} + 2\eta\psi = 0$$

gives $\psi = C_1 \exp(-\eta^2)$

Standard form of eqn.

$$\psi = \frac{d\phi}{d\eta} = C_1 \exp(-\eta^2)$$

thus

$$\phi = C_1 \int_0^{\eta} \exp(-\bar{\eta}^2)\, d\bar{\eta} + C_2$$

$\bar{\eta}$ variable of integration

Using the B.C.

$$\phi(\eta) = 1 - \frac{\int_0^{\eta} \exp(-\bar{\eta}^2)\, d\bar{\eta}}{\int_0^{\infty} \exp(-\bar{\eta}^2)\, d\bar{\eta}}$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} \exp(-\bar{\eta}^2)\, d\bar{\eta}$$

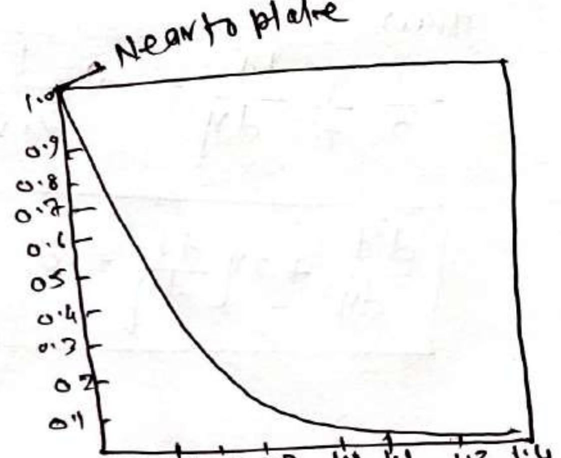
$$= 1 - \text{erf}(\eta)$$

error function, for detail refer to appendix C.6 of Bird.

$$\phi_\eta = \frac{v_x(y,t)}{v_0} = 1 - \text{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)$$

$\eta \rightarrow$ called complementary error function

Note erf(x) has value $= 0.99$ when $x = 2$



Notice that for $\eta = 2.0$ $\frac{v_x}{v_0} = 0.01$

which is the extent of Boundary layer penetration

therefore $y = \delta$

$$2.0 = \frac{y}{\sqrt{4\nu t}}$$

$$\therefore \boxed{\delta = 4\sqrt{\nu t}}$$

Chapter-6 Bird

Friction factor and equations for flow around object

~~Definition of friction factor~~:

Broadly there are two categories of flow

(a) Flow in channel → pipe flow, Pumping oil through pipes, water flow in open channel

(b) → Flow around an object
- → air around a plane
- → sedimentation
- → Flow across tube banks in HE

For class I → relationship b/w vol. flow rate & pressure drop is important

For class II → relation b/w the approaching velocity of the fluid and the drag force on the object

Not always easy to find velocity and p

relationship particularly if the flow is too turbulent or the geometry is complicated

For such system experiment are performed and correlation of dimension less variables are constructed

Friction Factor:

Consider the steady flow of a constant ρ fluid (a) in a conduit (b) around a submerged object that has an axis of symmetry or two planes of symmetry

Two types of forces will be exerted on the ~~fluid~~ solid surface by the fluid

(a) $F_s \longrightarrow$ Force even when the fluid is stationary

(b) $F_k \rightarrow$ Force when fluid is in motion

$F_k \propto A$ (characteristic Area)

$\propto K$ (characteristic kinetic energy (vol.) dynamic pressure.

$F_k = A \cdot K \cdot f \longrightarrow$ friction factor

hence to define $f \rightarrow$ need to define A & K

Case I Flow in conduits:

A → wetted surface ($2\pi RL$) → surface area because friction will occure at the surface

K → $\frac{1}{2} \rho V_{av}^2$ → $\frac{K.E}{Volume}$

$$F_K = 2\pi RL \left(\frac{1}{2}\rho V_{av}^2\right) \cdot f$$

force balance in the direction of flow is given

$$\therefore F_K = \left[(P_0 - P_1) + \rho g (h_0 - h_L)\right] \cdot \pi R^2$$

"Specially for circular tube of radius R and length 'L'" ($F_K = \tau_{rz} \cdot 2\pi RL = \left(\frac{P_0 - P_L}{2L} \cdot R\right) \cdot 2\pi RL$) 0 → L elevation

$$= (P_0 - P_L)\, \pi R^2$$

$$f = \frac{1}{4}\left(\frac{D}{L}\right)\left(\frac{P_0 - P_L}{\frac{1}{2}\rho \langle V\rangle^2}\right)$$

fanning friction factor | John Thomas fanning a civil Engr. (1837–1911)

$f_D = 4f$ → Darcy's friction factor (Mostly used by civil & Mechanical Engrs.)

Case II flow around a submerged object

Here we are considering the falling of sphere into liquid.

measured through (proof)

$$F_K = \text{gravitation force} - \text{Buyoncy force}$$

$$= \frac{4}{3}\pi R^3 \rho_s g - \frac{4}{3}\pi R^3 \rho_l \cdot g$$

$$= \frac{4}{3}\pi R^3 \cdot g\,(\rho_s - \rho_L)$$ This applies at steady state

Also for a ~~falling sphere~~ (A is usually the projected area in this case)

$$F_K = (\pi R^2)\left(\frac{1}{2}\rho_L V_\infty^2\right) \cdot f$$

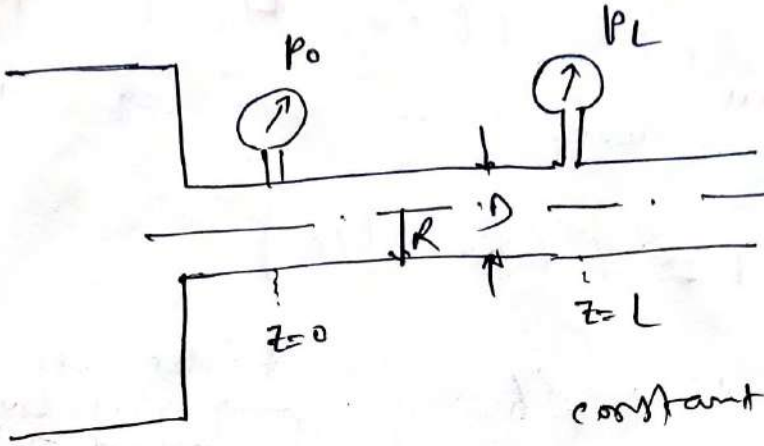
V_∞ → approach velocity at a large distance from the object

drag coeff. ← $f = \frac{4}{3} \frac{gD}{V_\infty^2}\left(\frac{\rho_s - \rho_l}{\rho_l}\right)$

(71)

To find the dependency of f on hydrodynamic property we need a non dimensional eqn. So let's examine

Friction factors for flow in Tubes.



constant ρ, & μ

Consider steady state flow Laminar / turbulent. Therefore force in z direction at the inner wall of the tube in z direction is

$$F_K(t) = \int_0^L \int_0^{2\pi} \left(-\mu \frac{\partial v_z}{\partial r}\right)\Big|_{r=R} R\,d\theta\,dz$$

time

in turbulent flow F_K may be a function of time due to turbulent fluctuations. In laminar flow F_K is independent of time

$$f(t) = \frac{\int_0^L \int_0^{2\pi} \left(-\mu \frac{\partial v_z}{\partial r}\right)\Big|_{r=R} R\,d\theta.\,dz}{2\pi R L \left(\frac{1}{2}\rho \langle v_z\rangle^2\right)}$$

Let us define $r^* = r/D$, $z^* = z/D$, $v_z^* = \frac{v_z}{\langle v_z\rangle}$

$t^* = \frac{\langle v_z\rangle . t}{D}$, $P^* = \frac{P - P_0}{\rho \langle v_z\rangle^2}$ & $Re = \frac{D\langle v_z\rangle \rho}{\mu}$

$$f(t^*) = \frac{1}{\pi}\frac{D}{L}\frac{1}{Re}\int_0^{L/D}\int_0^{2\pi}\left(-\frac{\partial v_z^*}{\partial r^*}\right)_{r^*=\frac{1}{2}} d\theta.\,dz^*$$

Valid for Laminar / Turbulent flow in circular tubes

*

$$\tilde{f}(\check{t}) * Re = \frac{1}{\pi} \cdot \frac{D}{L} \cdot \int_0^{L/D} \int_0^{2\pi} \left(-\frac{\partial \check{v}_z}{\partial \check{r}}\right)\Bigg|_{\check{r}=\check{r}_2} d\theta \, d\check{z}$$ (72)

solution of these eqn with appropriate B.C.s & initial condition leads that to

Actually $\check{V} = \check{V}(\check{r}, \check{\theta}, \check{z}, \check{t}; Re)$, $\check{p} = \check{p}(r, \theta, z, t; Re)$

$$f(\check{t}) = f(Re, L/D, \check{t})$$

if we use time avg of them

L.H.S is equal to the dimensionless avg. velocity gradient over the surface

$$f \equiv f(Re, L/D)$$

Further if the tube is sufficiently long

for flow systems in which drag depends upon on viscous force alone

$$\boxed{f = f(Re)}$$

→ This simplifies that we need only one curve for combination of $\frac{D\langle V\rangle \rho}{\mu}$ which is much easier than plotting pressure vs. flowrate for separate $D, \Delta P, L, \rho, \mu$,

Le = entrance region length

$Le = 0.03 D\, Re$ (Laminar flow)

$Le \simeq 60D$ (for turbulent flow)

For laminar flow

$$f = 1/4 \, (D/L) \left(\frac{P_0 - P_L}{\frac{1}{2}\rho\langle V\rangle^2}\right) \rightarrow \text{eq}^n \; (A)$$

$P_0 - P_L$ → from Hagen-Poiseuille eqn for laminar flow

$$(P_0 - P_L) = \frac{32 \mu \langle V\rangle L}{D^2}$$

$$\frac{kg\text{-}m}{m^2\text{-}s} \cdot \frac{m}{s} \cdot m \Rightarrow \dots$$

$$f = \frac{1}{4} \frac{D}{L} \left[\frac{32 \mu \langle V\rangle L}{D^2 \frac{1}{2}\rho\langle V\rangle^2}\right]$$

$$= \frac{16 \mu}{D\langle V\rangle \rho} = \frac{16}{Re}$$

$$\boxed{f = \frac{16}{Re}}$$ ✓

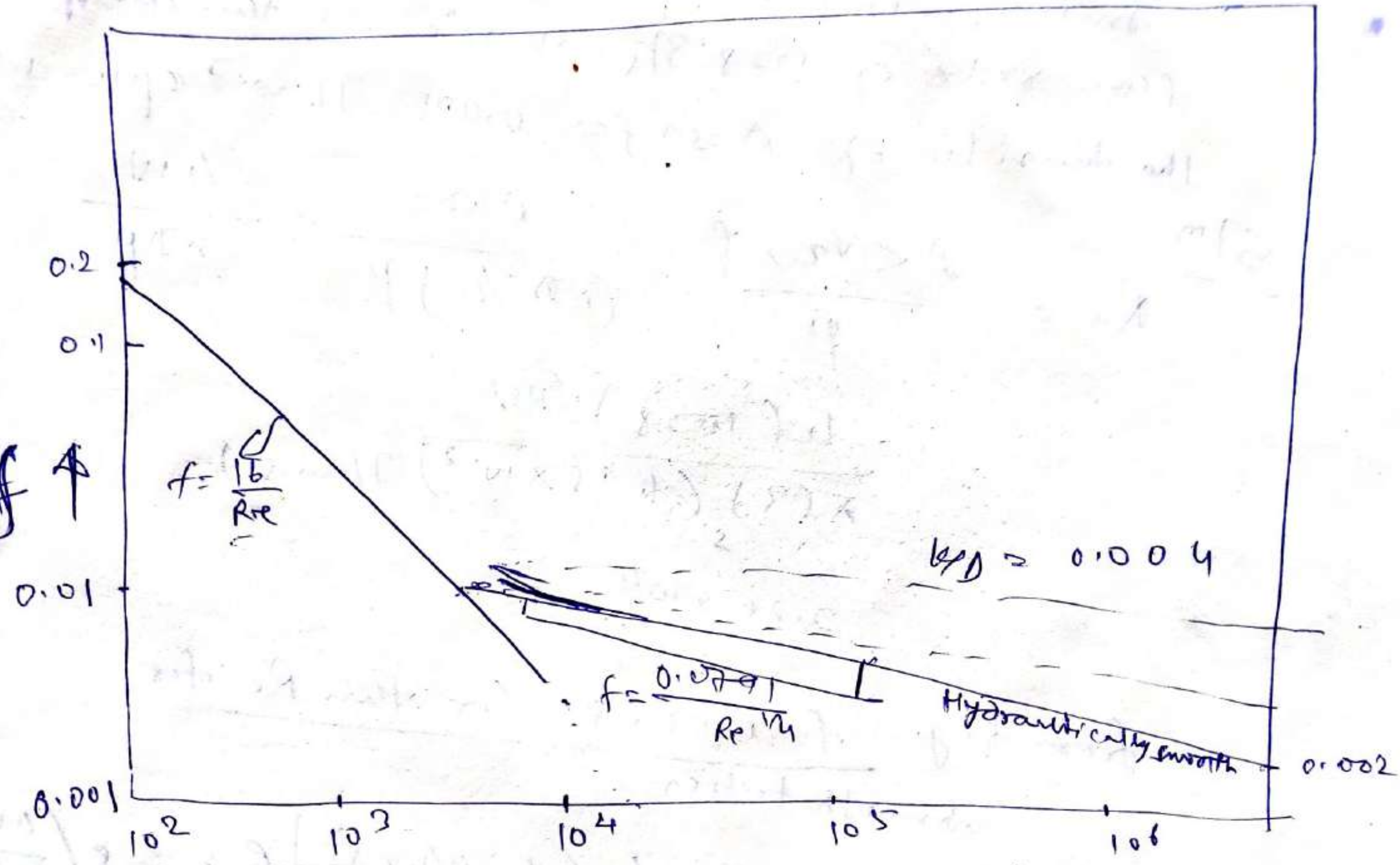
fanning factor

Calculate f using pressure & flowrate data using eqn (A) and then calculate Re and generate graph.

For turbulent flow → Blasius formula

$$f = \frac{0.0791}{Re^{(1/4)}} \qquad 2.1\times10^{3} < Re < 10^{5}$$

For smooth long circular tube.



$Re = D\langle v\rangle\rho/\mu$

$$\frac{1}{\sqrt{f}} = 4.0 \log_{10} Re\sqrt{f} - 0.4 \qquad 2.3\times10^{3} < Re < 4\times10^{6}$$

Prandtl's formula

For rough pipes (dashed curves)

$$\frac{1}{\sqrt{f}} = -3.6\log_{10}\left[\frac{6.9}{Re} + \frac{(k/D)^{10/9}}{3.7}\right] \left\{\begin{matrix} 4\times10^{4} < Re < 10^{8} \\ 0 < k/D < 0.05 \end{matrix}\right\}$$

Q. 6.2.1

What pressure gradient is reqd. to cause compound A, to flow in a horizontal smooth, circular tube, D = 3 cm, at a mass flow rate of 1028 g/s at 20°C. At this temp. the density of A is $\rho = 0.935$ g/cm³ & $\mu = 1.95$ cp

Sol^n

$$Re = \frac{D \langle v_z \rangle \rho}{\mu} = \frac{Dw}{(\pi D^2/4)\mu} = \frac{4w}{\pi D \mu}$$

$$= \frac{4(1028)\ (\text{g/s})}{\pi (3)(1.95 \times 10^{-2})\ (\text{g/cm.s})}$$

$$= 2.24 \times 10^4$$

from fig $f = 0.0063$ for this Re for smooth tubes

$$\therefore \left(\frac{P_0 - P_L}{L}\right) = \left(\frac{4}{D}\right)\left(\frac{1}{2}\rho \langle v_z \rangle^2\right) f = \frac{2}{D}\rho\left(\frac{4w}{\pi D^2 \rho}\right)^2 f$$

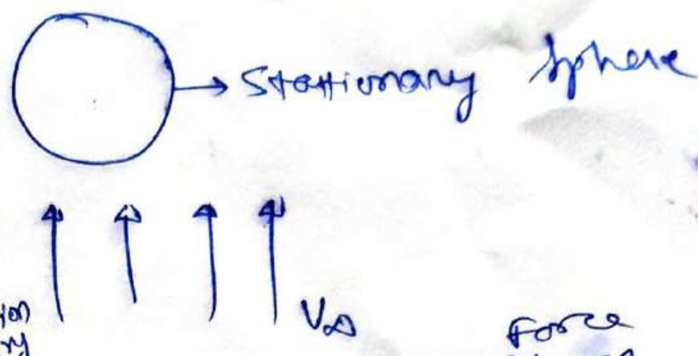
$$= \frac{32 w^2 f}{\pi^2 D^5 \rho} = \frac{32 (1028)^2 (0.0063)}{\pi^2 (3.0)^5 (0.935)}$$

$$= 95 \ (\text{dyne/cm}^2)/\text{cm}$$

$$= 0.071 \ (\text{mmHg/cm})$$

Cheque problem No. 6.2.2 Bird

Friction factor for flow around spheres.



$$F_K = (F_n - F_s) + F_t$$

(F_n: Normal ~~stress~~ force; F_s: ~~stress~~ force even when the fluid is stationary)

$$= F_{form} + F_{friction}$$

$F_n \rightarrow$ normal ~~stress~~ force

$F_t \rightarrow$ tangential force

$$f = f_{form} + f_{friction}$$

$$F_{form} = f_{form} \cdot A \cdot K, \quad F_{friction} = f_{friction} \cdot A \cdot K$$

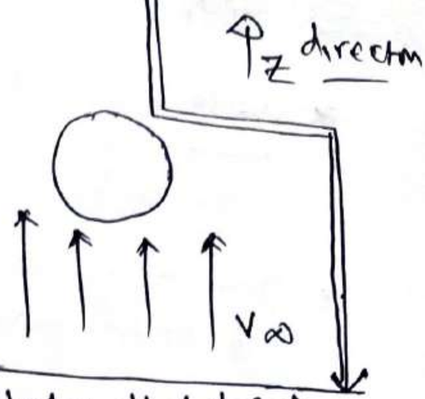
75a

Kinetic force $F_K = F_{form} + F_{friction}$

$(F_n - F_s)$ ⇙ ↳ F_t ← tangential force

$F_{form} \overset{\text{due to}}{\Rightarrow}$ Pressure force

$F_{friction} \Rightarrow$ Viscous forces



$$F_{form}(t) = \int_0^{2\pi}\int_0^{\pi}\left(-P\big|_{r=R}\cos\theta\right)R^2\sin\theta\, d\theta\, d\phi$$

and

$(v_r = 0)$, 0 at $r = R$

$$F_{friction}(t) = \int_0^{2\pi}\int_0^{\pi}\left(-\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\right]\Bigg|_{r=R}\sin\theta\right)R^2\sin\theta\, d\theta\, d\phi$$

Note that total force acting in z direction will be
$= F_n + F_t$
However F_s part of F_n remains present even when the fluid is stationary. Thus
$F_K = (F_n - F_s) + F_t$

$f = f_{form} + f_{friction}$

$$f_{form}(\breve{t}) = \frac{2}{\pi}\int_0^{2\pi}\int_0^{\pi}\left(-\breve{P}\big|_{\breve{r}=1}\cos\theta\right)\sin\theta\, d\theta\, d\phi$$

$$f_{friction}(\breve{t}) = -\frac{4}{\pi}\frac{1}{Re}\int_0^{2\pi}\int_0^{\pi}\left[\breve{r}\frac{\partial}{\partial \breve{r}}\left(\frac{\breve{v}_\theta}{\breve{r}}\right)\right]\Bigg|_{\breve{r}=1}\sin^2 d\theta\, d\phi$$

The dimensionless variables are

$$\breve{P} = \frac{P}{\rho V_\infty^2};\quad \breve{v}_\theta = \frac{v_\theta}{v_\infty};\quad \breve{r} = \frac{r}{R};\quad \breve{t} = \frac{v_\infty t}{R}$$

Reynolds number $Re = \dfrac{D v_\infty \rho}{\mu}$

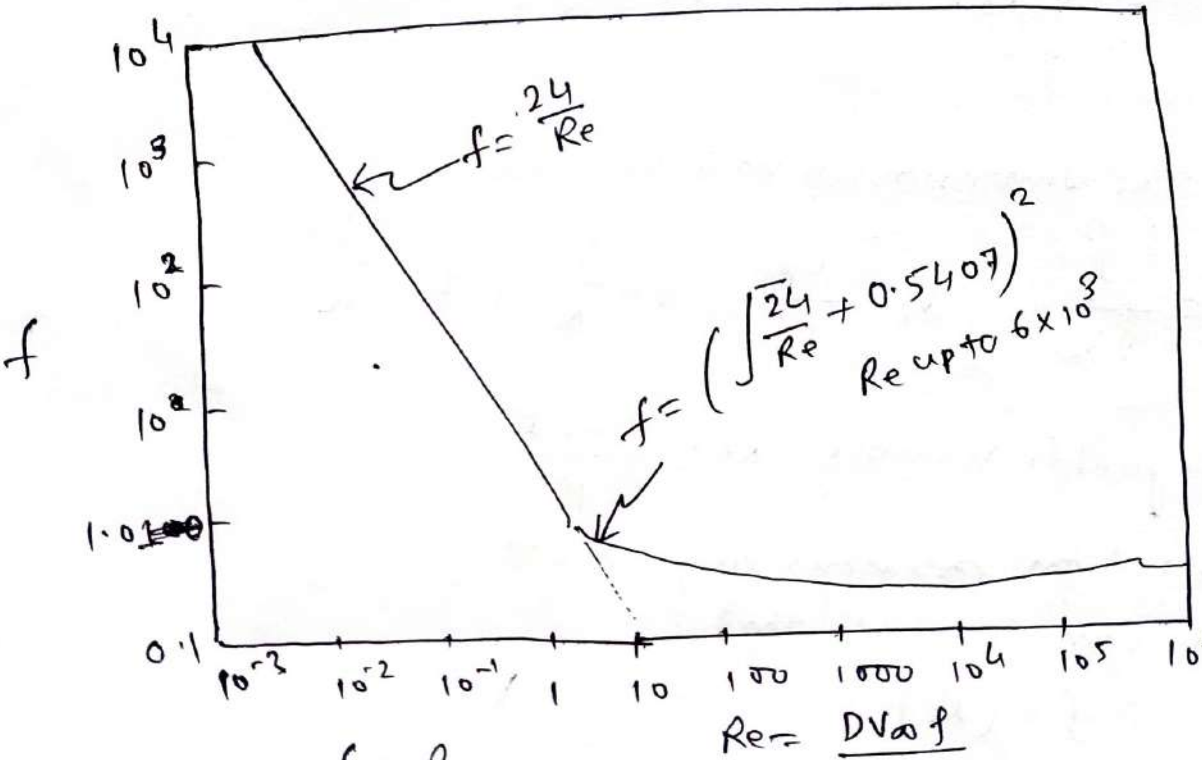
Based on previous arguments we can conclude that

$f = (Re)$

75b

Comparison

Flow in tubes	Flow around Sphere
* Well defined transition ~~and~~ from laminar to turbulent at about $Re = 2100$	No well defined transition
* The only contribution to f is friction	f = form friction + Skin friction
* No boundary layer Separation	There is a kink in f vs Re curve associated with a shift in the Separation zone



f vs Re

Friction factor (drag coefficient) for spheres moving relative to a fluid with velocity V_∞.

Note ⇒ $f \simeq 0.44$ for $5 \times 10^2 < Re < 1 \times 10^5$
(Known as Newton's law of Resistance)

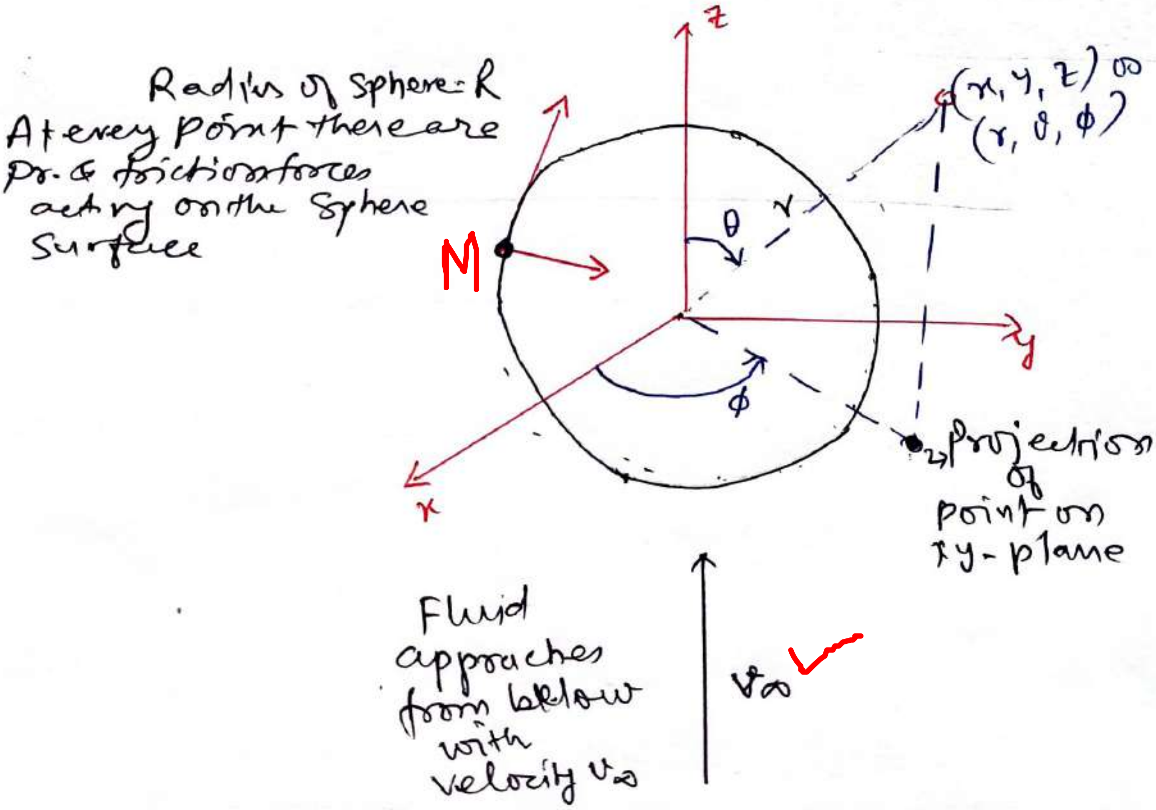
For the creeping flow region we already know that the drag force is given by Stokes' law which is consequence of solving eqn of continuity & eqn of motion without $\rho \frac{Dv}{Dt}$ term)

(See secth 2.6, Bird on creeping flow)

From there it is noted that

$$F^{(n)} = \underbrace{\frac{4}{3}\pi R^3 \rho g}_{\text{Buyoncy force part}} + \underbrace{2\pi \mu R v_\infty}_{\text{form drag Part}}$$

normal forces are obtained by integration of $(-p + \tau_{rr})\Big|_{r=R}$



At each point on the surface of the sphere the fluid exerts a force/unit area on the solid

$= F^{(n)} = -(p+\tau_{rr})\big|_{r=R}$

which acts normal to the surface

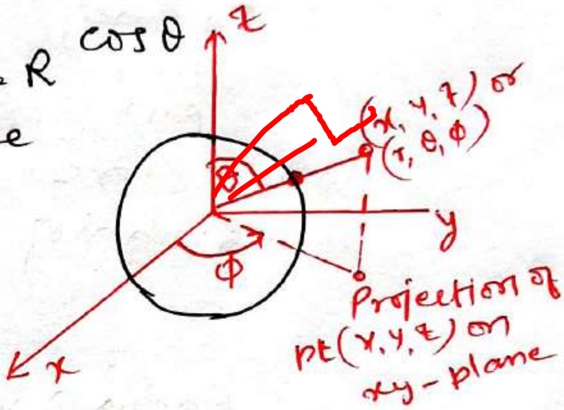
(−)ive sign is kept, as the sphere is in the region of lesser 'r' and ~~sphere~~ fluid in the region of higher 'r'

From Integration of normal force: - [from Bird sectn 2.6]

The z component of force in the direction of flow is $= -(p+\tau_{rr})\big|_{r=R}\cos\theta$

Now to get the normal force on the surface element let's multiply eq. by a differential surface element $R^2\sin\theta\, d\theta\, d\phi$, which is ⊥ to the r direction (see Fig A8.2 Bird)

Then integrate over the surface of the sphere to get the resultant normal force in z-direction:



$$F^{(n)} = \int_0^{2\pi}\int_0^{\pi} -\left((p+\tau_{rr})\big|_{r=R}\cos\theta\right) R^2\sin\theta\, d\theta\cdot d\phi$$

($\tau_{rr} = 0\ (\because v_r = 0)\big|_{r=R}$)

where p_0 is pressure far away from sphere in the plane $z=0$:

$$p\big|_{r=R} = p_0 - \rho g R\cos\theta - \frac{3}{2}\frac{\mu v_\infty}{R}\cos\theta$$

$$\tau_{rr} = -2\tau_{\theta\theta} = -2\tau_{\phi\phi} = \frac{3\mu v_\infty}{R}\left[-\left(\frac{R}{r}\right)^2 + \left(\frac{R}{r}\right)^4\right]$$

$$\tau_{r\theta} = \tau_{\theta r} = \frac{3}{2}\frac{\mu v_\infty}{R}\left(\frac{R}{r}\right)^4\sin\theta$$

Note: that normal stress for this flow ($\tau_{rr}, \tau_{\theta\theta}, \tau_{\phi\phi}$) are non zero except at $r=R$ i.e. at sphere surface where the velocity of fluid is zero. Further, because of the symmetry around z-axis the resultant force will be in z direction.

Substitution of p in $F^{(n)}$ eqn yields on integration.

that $p_0 \longrightarrow 0$

ρg term $\longrightarrow$ Buoyancy force

μV_∞ term $\longrightarrow$ form drag.

~~likewise~~ So $F^{(n)}$ becomes

$$F^{(n)} = \frac{4}{3}\pi R^3 \rho g + 2\pi\mu R V_\infty$$

$2\pi\mu R V_\infty$ → Form drag

Similarly Integration of tangential force gives.

$$F^{(t)} = \int_0^{2\pi}\int_0^{\pi}\left(\tau_{r\theta}|_{r=R}\sin\theta\right)R^2\sin\theta\, d\theta\, d\phi$$

$\tau_{r\theta}|_{r=R}\sin\theta$ → z-component of the tangential force/area

$R^2\sin\theta\, d\theta\, d\phi$ → elemental area.

Since $\tau_{r\theta}|_{r=R} = \frac{3}{2}\frac{\mu V_\infty}{R}\sin\theta$

$$\therefore F^{(t)} = 4\pi\mu R V_\infty$$

→ frictional drag

The total force thus becomes

$$F = F^{(n)} + F^{(t)}$$

$$= \frac{4}{3}\pi R^3 \rho g + 2\pi\mu R V_\infty + 4\pi\mu R V_\infty$$

Buoyant force, form drag, friction drag

$$= F_b + F_k$$

$$\boxed{F_k = 6\pi\mu R V_\infty}$$

STOKES' LAW for $Re < 0.1$

Rearranging

$$F_K = 6\pi\mu R V_\infty = A \cdot K \cdot f$$

$$= \frac{6\pi R^2}{R} \cdot \mu V_\infty \cdot \frac{\frac{1}{2}\rho V_\infty^2}{\frac{1}{2}\rho V_\infty^2}$$

$$= \frac{24\pi R^2}{D} \cdot \frac{\mu V_\infty}{(\rho V_\infty^2)} \cdot \left(\tfrac{1}{2}\rho V_\infty^2\right)$$

$$= \frac{24\pi R^2}{D} \cdot \left(\frac{\mu}{\rho V_\infty}\right) \cdot \left(\tfrac{1}{2}\rho V_\infty^2\right)$$

$$= \frac{24}{\left(\frac{D V_\infty \rho}{\mu}\right)} \cdot (\pi R^2) \cdot \left(\tfrac{1}{2}\rho V_\infty^2\right)$$

$$= \underline{\underline{f}} \quad \underline{\underline{A}} \quad \underline{\underline{K}}$$

$$\therefore \boxed{f = \frac{24}{Re}} \qquad Re < 0.1$$

for creeping flow

Another relation

$$f = \left(\sqrt{\frac{24}{Re}} + 0.5407\right)^2 \qquad Re < 6000$$

&

$$f = 0.44 \qquad 5 \times 10^2 < Re < 10^5$$

↳ Newton's law of Resistance: According to this law drag force is ∝ to the square of approach velocity

Q. Determination of the diameter of a falling sphere glass sphere $\rho_{sph} = 2.62\ g/cm^3$, cely at $20^\circ C$ $\rho \rightarrow 1.59\ g/cm^3$ $\mu \rightarrow 9.58$ milipoise. Find the dia of the sphere to have $V_\infty = 65$ cm/s.

Consider the eqn for flow around a sphere

$$f = \frac{4}{3}\frac{Dg}{V_\infty^2}\cdot\left(\frac{\rho_s - \rho_l}{\rho_l}\right)$$

Since we have f vs Re curve, so let's rearrange the above eqn.

$$\frac{f}{Re} = \frac{4}{3}\frac{g\mu}{\rho V_\infty^2}\left(\frac{\rho_s - \rho_l}{\rho_l}\right) \quad ; Re = \frac{D V_\infty \rho}{\mu}$$

$$= C = \frac{4}{3} \times \frac{(980)(9.58 \times 10^{-3})}{(1.59)(65)^3}\left(\frac{2.62 - 1.59}{1.59}\right)$$

$$= 1.86 \times 10^{-5} \Rightarrow \text{dimensionless}$$

Find f from the f vs Re curve for sphere which gives

$$\frac{f}{Re} = \frac{1.86}{10^5} \Rightarrow 1.86 \times 10^{-5}$$

draw a line of slope = 1

from the poin $f \rightarrow 1.86$

$Re \rightarrow 10^5$

UNIT- IV

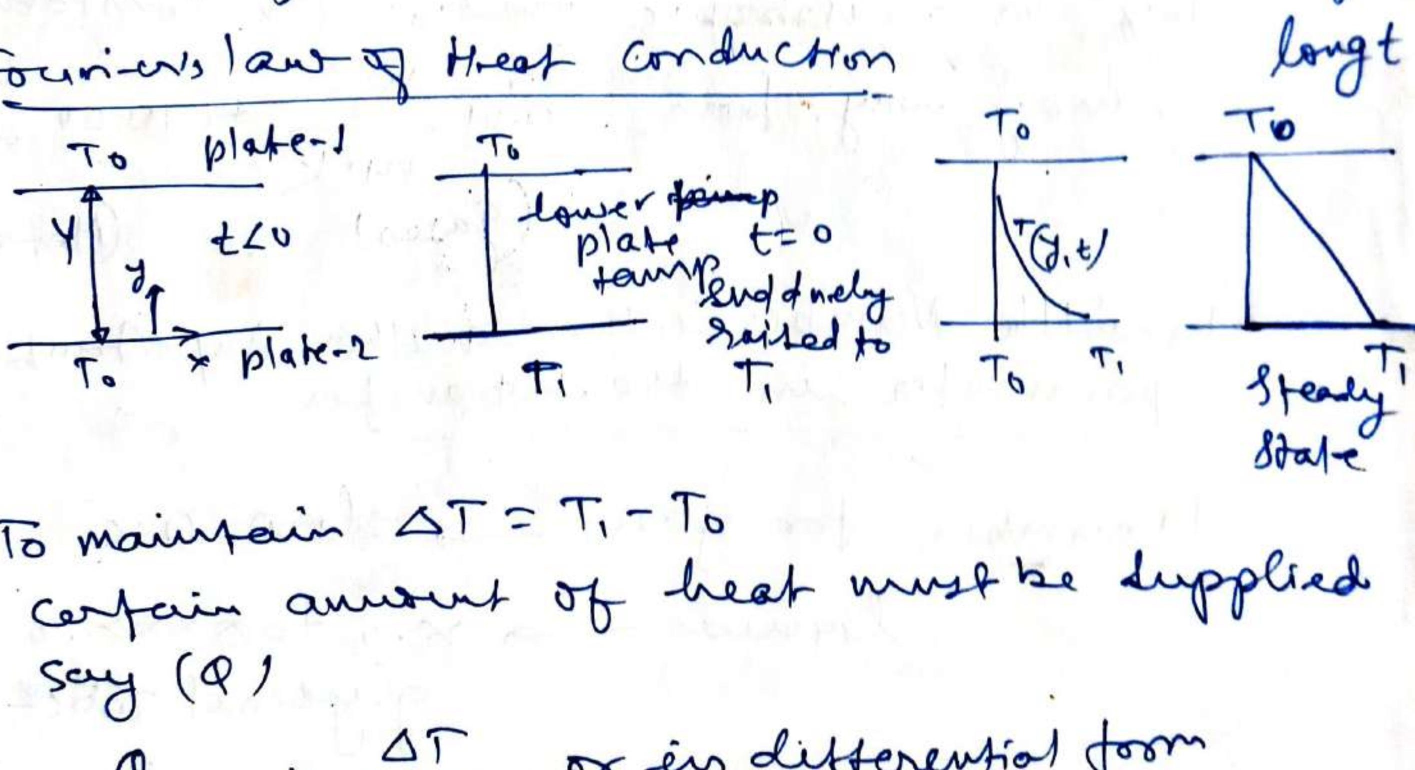
Energy Transport Phenomena

Energy Transport
- → Conduction (Molecular energy transport position of molecules are kind of fixed)
- → Convection (Bulk motion for the fluid)
  - (i) In convection at molecular level the transfer is still through conduction.
  - (ii) In convection molecules changes its position.
- → Diffusive transport → inter diffusing mixture
- → Radiation → No medium is reqd.

Thermal conductivity describes at what rate heat is conducted in a material

~~H.T. in fluid~~

Fourier's law of Heat Conduction



To maintain $\Delta T = T_1 - T_0$ certain amount of heat must be supplied say (Q)

$$\frac{Q}{A} = -k \frac{\Delta T}{y}$$ or in differential form

$$\boxed{q_y = -k \frac{dT}{dy}}$$

Heat flow (+)ive quantity
Flow in (+)ive y direction

For temperature variation in three dimensional form.

$$q_x = -K_x \frac{\partial T}{\partial x} \quad \text{——} \; (1)$$

$$q_y = -k_y \frac{\partial T}{\partial y} \quad \text{——} \; (2)$$

$$q_z = -k_z \frac{\partial T}{\partial z} \quad \text{——} \; (3)$$

$$\bar{q} = \delta_x \cdot q_x + \delta_y \cdot q_y + \delta_z \cdot q_z$$

$$\boxed{\bar{q} = -K \, \nabla T}$$

Three dimentional form of Fourier's law.

It describes molecular transport of heat by in a isotropic media (k constant)

k - may vary from $0.01 \frac{W}{m \cdot k}$ (gases) to $1000 \frac{W}{m \cdot k}$ (Metals)

Prandtl Number is the another important parameter in Heat transfer

Pr number for gases ⟶ low 0.x

liquids ⟶ x.0 to xxxx.0

glycerol 30°C - 6580

350°C - 329

Energy is a scalar
Momentum is a vector

(56)

Pr → liquid metals — very low 0.00XX to 0.0XX
Na, K, Hg Pb, Bi

(See liquid metals handbook)
with T → may be high/low (see 9.1-4 Bird)

k of solids → with temp. k may be high or low

Ex Al. 373.2 → 205.9
573.2 → 268 873.2 → 423

Cd 273.2 → 93.0
373.2 → 90.4

wood parallel to axis → 0.126 | anisotropic
normal to axis → 0.038 |

Temperature & pressure dependence of Thermal conductivity:-

~~You may refer to the monograph~~ in fig 9.2-1

* for gases k↑ with T↑, for liquid k↓ with T↑

Chapman enskog formula for k of a monoatomic gas of low density & T

REFER Sect 9.5 Bird

$$k = 1.9891\times10^{-4}\,\frac{\sqrt{T/M}}{\sigma^2\Omega_k}$$

by comparing with similar viscosity formula $\mu = 2.6693\times10^{-5}\,\frac{\sqrt{MT}}{\sigma^2\Omega_\mu}$

k → $\frac{Cal}{cm.s.K}$
T → K
σ → Å (collision diameter)
Ω_k → collision integral for Lennard-Jones potential
Thermal conductivity Table E.2

eqn is good for monoatomic gas say Ne, Xe, Ar, etc.

for polyatomic gas at low density

$$k = \left(\hat{C}_p + \frac{5}{4}\frac{R}{M}\right)\mu$$ Eucken's formula (Ot-ken's)

^ → circumflex for per/mass
~ → Tilde for per mole

as a special case $\hat{C}_p = 5/2\ R/M$ → for Monoatomic gas.

$$\left(k = \frac{15}{4}\frac{R}{M}\mu = \frac{5}{2}\hat{C}_v\mu\right.$$ for monoatomic gas)

For mix. of gas at low density

$$k_{mix} = \sum_{\alpha=1}^{N} \frac{x_\alpha k_\alpha}{\sum_\beta x_\beta \phi_{\alpha\beta}}$$

$x_\alpha \rightarrow$ mole fraction
$k_\alpha \rightarrow$ The cond. of pure gas
$\phi_{\alpha\beta} \rightarrow$ constant

Q. 9.3-1

Compute the thermal conductivity of a monoatomic gas of low density

For Ne → Parameter (Leonard-Jones) from Table E.1

$\sigma = 2.789\,A^\circ$, $\epsilon/k = 35.7\,K$, $M = 20.183$ unit. wt.

(ϵ = characteristic energy)

∴ at $373\,K$ $\quad kT/\epsilon = \frac{373.2}{35.7} = 10.45$

from Table E.2 $\quad \Omega_k = \Omega_\mu = 0.821$

Now $\quad k = 1.981 \times 10^{-4} \frac{(T/M)^{1/2}}{\sigma^2 \Omega_k}$

$$= 1.981 \times 10^{-4} \frac{(373.2/20.1)^{1/2}}{(2.789)^2\,(0.821)}$$

$$= 1.338 \times 10^{-4} \frac{cal}{cm \cdot s \cdot K}$$

Measured value $= 1.37 \times 10^{-4} \frac{cal}{s \cdot cm \cdot K}$

9.3-2

Estimate the thermal conductivity of molecular oxygen at 300K and low pressure

(Th. conductivity of polyatomic gases low density)

Mol. wr. of O_2 = 32.0 $\tilde{C}_{p,300K} = 7.019 \frac{cal}{g.mol.K}$

from Table E.1 Leonard Jones parameter for molecular oxygen to be

$\sigma = 3.433 A^{\circ}$ and $\epsilon/\kappa = 113 K$

At 300K then $\kappa T/\epsilon = \frac{300}{113} = 2.655$

Table E.2 $\Omega_\mu = 1.074$ The viscosity

from Eq. 1.4-18

$$\mu = 2.6693 \times 10^{-5} \frac{\sqrt{MT}}{\sigma^2 \Omega_\mu}$$

$$= 2.6693 \times 10^{-5} \frac{\sqrt{32.00 \times 300}}{(3.433)^2 (1.074)}$$

$$= 2.065 \times 10^{-4} \text{ g/cm.s}$$

from Eucken approximation

$$k = \left(\tilde{C}_p + \frac{5}{4} R\right)\left(\mu / M\right)$$

$$= (7.079 + 2.484)(2.065 \times 10^{-4}) / 32.00$$

$$= 6.14 \times 10^{-5} \frac{cal}{cm.s.K}$$

9.3-3 do your self.

* Thermal conductivity of liquids

$$k = 2.80 \left(\frac{\tilde{N}}{\tilde{V}}\right)^{2/3} K v_s$$ (modified Bridgman's formula)

$\left(\frac{\tilde{V}}{\tilde{N}}\right)$ → volume / molecule

modification 3 → 2.80

eq^n applicable to low densities well above critical density

- Sonic velocity → v_s

$\frac{\tilde{V}}{\tilde{N}}$ → volume / molecule

K → Boltzmann constant

The velocity of low frequency sound

$$v_s = \sqrt{\frac{C_p}{C_v}\left(\frac{\partial p}{\partial \rho}\right)_T}$$

$\left(\frac{\partial p}{\partial \rho}\right)_T$ → may be obtained from eq^n of state

$\left(\frac{C_p}{C_v}\right)$ → 1 for liquids except near critical point

Prediction of the thermal conductivity of a liquid

9.4-1 ~~Do it yourself~~. The density of liquid CCl_4 at 20°C and 1 atm is 1.595 g/cm3, and its isothermal compressibility $\frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T = 90.7 \times 10^{-6}$ atm^{-1}. What is the thermal conductivity?

sol^n $$\left(\frac{\partial p}{\partial \rho}\right)_T = \frac{1}{\rho\left(\frac{1}{\rho}\right)\left(\frac{\partial \rho}{\partial p}\right)_T} = \frac{1}{1.595 \times 90.7\times10^{-6}} = 6.91\times10^{3} \frac{\text{atm cm}^3}{g}$$

$$6.91\times10^{3} \times 1.0133\times10^{6} \frac{g}{cm\,s^2}\cdot\frac{cm^3}{g} = 7.00\times10^{9} \frac{cm^2}{s^2}$$

Assuming $\frac{C_p}{C_v} = 1.0$ (for liquids) → $v_s = \sqrt{\frac{C_p}{C_v}\left(\frac{\partial p}{\partial \rho}\right)_T} = 8.37\times10^{4} \frac{cm}{s}$

molar volume $\tilde{V} = \frac{M}{\rho} = \frac{153.84}{1.595} = 96.5 \frac{cm^3}{g\,mole}$

from $k = 2.80\left(\frac{N}{V}\right)^{2/3} K v_s = 2.80\left(\frac{6.023\times10^{23}}{8.37\times10^{21}}\right)^{2/3}\frac{(1.3815\times10^{-16})}{(8.37\times10^{4})}$

$= 1.10\times10^{4} (cm^{-2}) / \left(\frac{erg}{k}\right)(cm/s)$

$= 0.110 \frac{W}{m\cdot K}$

Thermal conductivity of solids:

Th.C. (solid) should be measured experimentally as they depend on many factors (porosity, orientation, fluid contained in the pores)

Heat conduction. ~~Electricity conduction~~

~~Pure metal~~ ~~good~~

Pure metals better heat conductors than nonmetals. Crystalline metals conduct heat more readily than amorphous materials. Dry porous poor heat conductors. ~~and are therefore~~

	Temp ↑	The cond. ↑
Metals	↑	———nf ↓
Nonmetals	———nf	
Alloys	↑	intermediate behaviours

relⁿ with electrical thermal conductivity for pure metal (Lorenz eqⁿ)

$$\frac{k}{k_e T} = L$$

$k_e \rightarrow$ electrical Th. C.

$L \rightarrow$ Lorenz number $\simeq 22 - 29\times10^{-9} \frac{volt^2}{k^2}$ for pure metals at $0°C$

"L" increases by 10-20% per 1000°C Typically.

At very low temp metals becomes superconductor of electricity hence L varies rapidly in the at low temp. region. (Superconducting region)

Assignments 9A1–9A8, 9A9, 9A10

Effective Thermal conductivity of solid

Solids with pores or solid dispersed in ~~other~~ another solid (Two phase solid).

It can be treated as a homogeneous material of thermal conductivity (K_{eff})

Convective Transport of Energy:

Transport due to bulk motion of fluid across the surface element ds ⊥ to the x axis is

$$= \underbrace{\left(\frac{1}{2}\rho v^2 + \rho \hat{U}\right)}_{\text{Energy}/\text{vol.}} \underbrace{v_x\, dS}_{\text{vol. flow}/s} \rightarrow \frac{\text{Energy}}{s}.$$

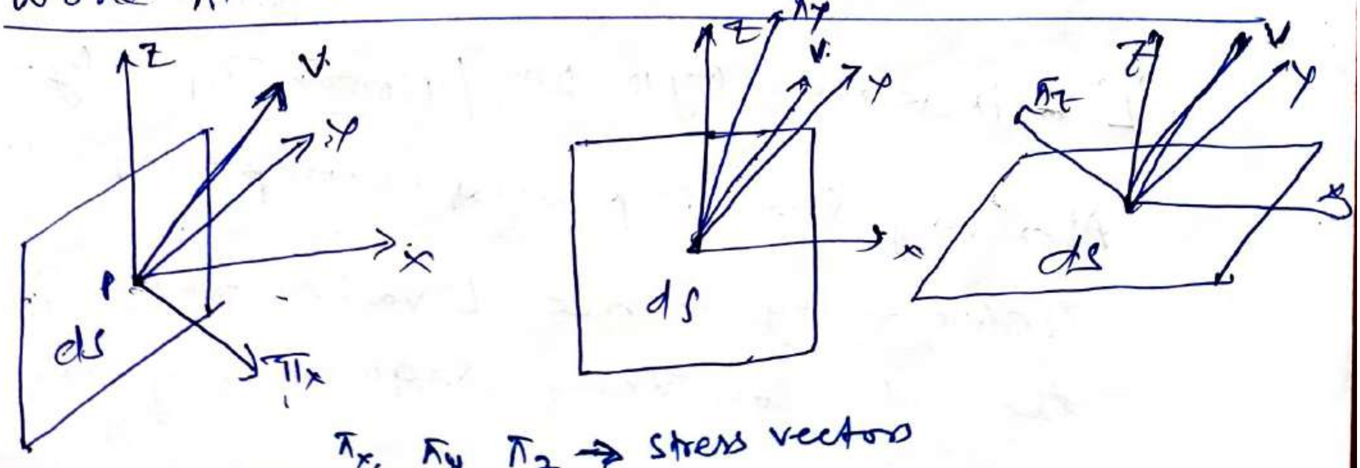
Note $\frac{1}{2}\rho v^2 = \frac{1}{2}\rho(v_x^2+v_y^2+v_z^2)$

convective energy flux $= \left(\frac{1}{2}\rho v^2+\rho\hat{U}\right)\delta_x v_x + \left(\frac{1}{2}\rho v^2+\rho\hat{U}\right)\delta_y v_y + \left(\frac{1}{2}\rho v^2+\rho\hat{U}\right)\delta_z v_z$

$$= \left(\frac{1}{2}\rho v^2+\rho\hat{U}\right)\mathbf{v}$$

is called convective energy flux vector. It is a flux from (−)ive side to (+)ive side

Work Associated with Molecular motion:



$\pi_x, \pi_y, \pi_z \rightarrow$ stress vectors

As the fluid is moving with velocity v

rate of work done by (−)ive fluid on (+) fluid is

Note: The fluid on the minus side of the surface exerts a force $\pi_x ds$ on the fluid that is on the (+)ive side.

$(\pi_x \cdot v)\, ds$

is the component form

$(\pi_x \cdot v) = \pi_{xx} v_x + \pi_{xy} v_y + \pi_{xz} v_z \equiv [\pi \cdot v]_x$

$(\pi_y \cdot v) = \pi_{yx} v_x + \pi_{yy} v_y + \pi_{yz} v_z \equiv [\pi \cdot v]_y$

$\pi \rightarrow$ Molecular stress tensor

$\pi = p\delta + \tau$

So that $[\pi \cdot v] = pv + [\tau \cdot v]$

molecular work flux vector: $\delta_x(\pi_x \cdot v) + \delta_y(\pi_y \cdot v) + \delta_z(\pi_z \cdot v)$

The term pv can then be combined with the internal energy term $\rho U v$ to give an enthalpy term

$\rho \hat{U} v + p v = \rho\,[\hat{U} + p/\rho]\, v$

$= \rho(\hat{U} + p\hat{V})\, v = \rho \hat{H} v$

$e = (\tfrac{1}{2}\rho v^2 + \rho \hat{U})\, v + [\pi \cdot v] + q$

or $e = \underbrace{(\tfrac{1}{2}\rho v^2 + \rho \hat{H})\, v}_{\text{convective energy flux}} + [\tau \cdot v] + q$ → molecular heat flux vector

e → Combined energy flux vector.

Enthalpy can be represented as a fn of temp by

$\hat{H} - \hat{H}_0 = \int_{T^0}^{T} \hat{C}_p\, dT + \int_{p^0}^{p} \left[\hat{V} - T\left(\frac{\partial \hat{V}}{\partial T}\right)_p \right] dp$

$\hat{H}_0$ → enthalpy/mass at the reference state.

integral over p $= 0$ (for ideal gas)

$= \frac{1}{\rho}(p - p^0)$ for fluids of const. ρ

(91.b)

The integral over T becomes

$C_p^a (T - T^0)$ if the heat capacity can be regarded as const. over the relevant temp. range.

Assignment 9A.1, 9A.2, 9A.3, 9A.4, 9A.5, 9A.8, 9A.10

Chapter 10 Bird.

Shell Energy Balance & Temp distribution in solids and Laminar flow.

General Energy Balance eqn at S.S.

rate of (~~Heat~~ (In − Out) convection + (In − Out) mol. transport + ~~(transport)~~)

(work done on syst by − work done by system) mol. transport + work done by External system + ~~rate~~ Energy prod] = 0

This is a I law of thermo written for open system.

Above eqn generates a ~~I. order~~ O.D.E for this slab to be solved with suitable B.C.

common B.C

(1) Specify the surface temp.

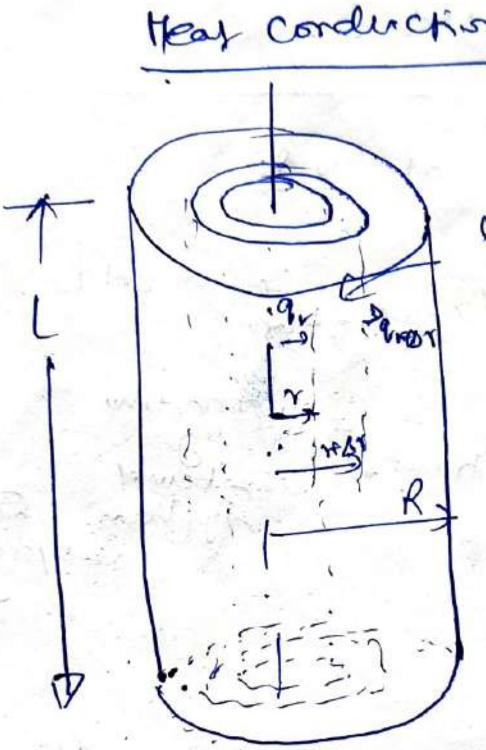
(2) heat flux normal to a surface may be given (as good as specifying the normal comp of the temp. gradient for temp.)

(3) ~~Temp~~ ~~continuity at the surface and heat flux~~ Temp & heat flux continuity at the surface

(4) at solid fluid surface

$q = h(T_0 - T_b)$ Newton's law of cooling

Heat conduction with an Electrical Source:



Uniform heat production by electrical heating S_e.

wire of Radius R and electrical conductivity k_e ohm^{-1} cm^{-1}

let rate of heat production

$$\boxed{S_e} = \frac{I^2}{k_e}$$

heat source due to electrical dissipation

Assuming that the temp rise is not so large

then k & $k_e \neq f(T)$

The surface of the wire is maintained at T_0

Q.2 Find the radial temp distributio

for energy balance consider a shell of thickness Δr and length L.

Since $v = 0$ eqn is

$$e = \{ (\rho v^2 + \rho \hat{H}) v + (\tau \cdot v) \} + q$$

$$\therefore e = q$$

$q_r \rightarrow$ heat flux molecular

$(q_r - q_{r+\Delta r})\, 2\pi r L$ $\Delta r \rightarrow$ very small

&

Rate of heat production = $(2\pi r \Delta r L) \cdot S_e$.

combine

$$(r q_r - r q_{r+\Delta r})\, 2\pi r L + 2\pi r \Delta r L \cdot S_e = 0$$

$$-\frac{d(q r)}{dr} + S_e r = 0$$

$$\frac{d(q r)}{dr} = S_e r \quad \text{or}$$

~~..., q_r is that~~

$$q r = \frac{S_e r^2}{2} + C_1$$

or

$$q = \frac{S_e r}{2} + \frac{C_1}{r}$$

at $r = 0$, q is finite $\therefore C_1 = 0$

$$\boxed{q_r = \left(\frac{S_e r}{2}\right)}$$

$$q_r = -k \frac{dT}{dr}$$

$$\therefore -k \frac{dT}{dr} = \frac{S_e r}{2}$$

at $r = R$, $T = T_0$ — B.C.

$$\therefore \boxed{T - T_0 = \frac{S_e R^2}{4k}\left[1 - \left(\frac{r}{R}\right)^2\right]}$$

It is a parabolic fn.

(95)

(i) Max temp rise

$$T_{max} - T_0 = \frac{S_e R^2}{4k}$$

(ii) Avg temp rise

$$\langle T \rangle - T_0 = \frac{\int_0^{2\pi}\int_0^R (T(r) - T_0)\, r\,dr\,d\theta}{\int_0^{2\pi}\int_0^R r\,dr\,d\theta} = \frac{S_e R^2}{8k}$$

$$= \frac{1}{2} T_{max}$$

(iii) Heat outflow at the surface (for a length of L of wire)

$$Q\Big|_{r=R} = 2\pi R L \cdot q_{r=R}$$

$$= 2\pi R L \cdot \frac{S_e R}{2} = S_e \cdot \pi R^2 L$$

clearly heat input = heat output at S.S

Comparison with Momentum transport

	Tube flow	Heated wire
First integration	$\tau_{rz}(r)$	$q_r(r)$
II^nd integration	$v_z(r)$	$T(r) - T_0$
B.C.s $r=0$	τ_{rz} = finite	q_r = finite
$r=R$	$v_z = 0$	$T - T_0 = 0$
Property (Transport)	μ	k
Source term.	$\left(\frac{P_0 - P_L}{L}\right)$	S_e
Assumption.	μ = const	k, k_e = const

Geankoplis 2nd (4.1D) - Check

Thermal conductivity of gas.

$k \propto (\sqrt{T})$ (Chapman eqn)

$k \neq f(P)$ but at very low pressure

$k \to 0$

Th. cond. of liquid: refer (Reid et al. 1977)

Energy is transferred due to mol. collision.

* Reid et al. → properties of gases and liquids Prausnitz

$k = a + bT \quad k \neq f(P)$

$k_{water} > k_{organic\,liqd.}$

Th. cond. of solid: varies quite widely

metals have very high Th. c.

wood, rock etc. have very low Th. c.

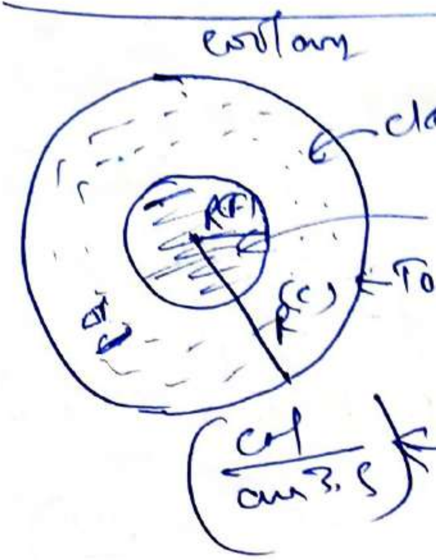
Two mechanism of H.T in solid

(1) heat is conducted by free electron

(2) Heat is conducted by translational energy.

Assignment | 10.2.1 example. | 10.2.2

Heat conduction with a Nuclear heat Source:



Energy generated from nuclear fission ~~$S_n = S_{n0}\left[1 + b\left(\frac{r}{R^{(F)}}\right)^2\right]$~~

$$S_n = S_{n0}\left[1 + \frac{b}{1}\left(\frac{r}{R^{(F)}}\right)^2\right]$$

S_{n0} → volume rate of heat production at the centre of the sphere. and b is a dimensionless (+)ive constant.

No flow hence $e = q$

consider a sphere of thickness Δr.

rate of Heat in: $q_r^{(F)}\big|_r \cdot 4\pi r^2$,

rate of heat Out: $q_r^{F}\big|_{r+\Delta r} \; 4\pi(r+\Delta r)^2 \simeq 4\pi r^2 q_r^{(F)}\big|_{r+\Delta r}$

Rate of thermal energy produced by nuclear fission: $S_n \cdot 4\pi r^2 \Delta r$

making a balance

$$\frac{d}{dr}\left(r^2 q_r^{(F)}\right) = S_n r^2 \qquad \Delta r \to 0$$

(97)

$$\frac{d}{dr}\left(r^2 q_r^{(F)}\right) = S_{n0}\left[1 + b\left(\frac{r}{R^{(F)}}\right)^2\right] r^2$$

For cladding

$$\frac{d}{dr}\left(r^2 q_r^{(C)}\right) = 0$$

On integration

$$q_r^{(F)} = S_{n0}\left[\frac{r}{3} + \frac{b}{R^{(F)2}}\cdot\frac{r^3}{5}\right) + \frac{C_1^{(F)}}{r^2}$$

$$q_r^{(C)} = \frac{C_1^{(C)}}{r^2}$$

From II nd B.C., at $r = R^{(F)}$ $q_r^{(F)} = q_r^{(C)}$

$\therefore q_r^{(F)}\big|_{R^{(F)}} = \frac{C_1^{(C)}}{R^{(F)2}}$

$S_{n0}\left[\frac{R^{(F)}}{3} + \frac{b}{R^{(F)2}}\cdot\frac{R^{(F)3}}{5}\right) = \frac{C_1^{(C)}}{R^{(F)2}}$

$\therefore C_1^{(C)} = S_{n0}\left[\frac{1}{3} + \frac{b}{5}\right] R^{(F)3}$

B.C.s

at $r = 0$, $q_r^{(F)}$ is finite

at $r = R^{(F)}$ $\quad q_r^{(F)} = q_r^{(C)}$ $\quad$ continuity of flux

$$q_r^{(F)} = S_{n0}\left(\frac{r}{3} + \frac{b}{R^{(F)2}}\cdot\frac{r^3}{5}\right)$$ Flux distribution fission able material

$$q_r^{(C)} = S_{n0}\left(\frac{1}{3} + \frac{b}{5}\right)\frac{R^{(F)3}}{r^2}$$ cladding

~~Substitute~~ Substitute Fourier's law to find temp distribution

$$-k^{(F)}\frac{dT^{(F)}}{dr} = S_{n0}\left(\frac{r}{3} + \frac{b}{R^{(F)2}}\cdot\frac{r^3}{5}\right)$$

$$-k^{(C)}\frac{dT^{(C)}}{dr} = S_{n0}\left(\frac{1}{3} + \frac{b}{5}\right)\frac{R^{(F)3}}{r^2}$$

(98)

$$T^{(F)} = -\frac{S_{n0}}{k^{(F)}}\left(\frac{r^2}{6} + \frac{b}{R^{(F)2}}\frac{r^4}{20}\right) + C_3^{(F)}$$

$$T^{(C)} = \frac{S_{n0}}{k^{(C)}}\left(\frac{1}{3} + \frac{b}{5}\right)\frac{R^{(F)3}}{r} + C_3^{(C)}$$

B.C.'s

at $r = R^{(F)}$ $\quad T^{(F)} = T^{(C)}$ $\quad$ continuity of temp

$r = R^{(C)}$ $\quad T^{(C)} = T_0$

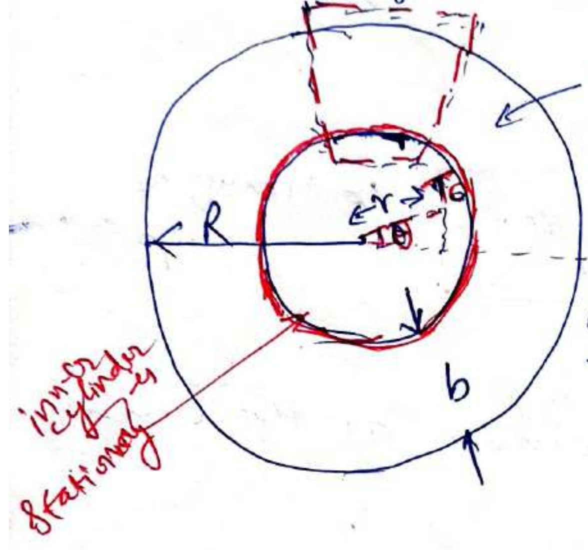
where T_0 is the known temperature at the outside of the cladding

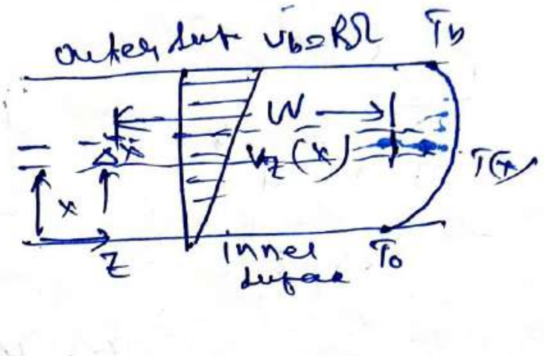
$$T^{(F)} = \frac{S_{n0}R^{(F)2}}{6k^{(F)}}\left\{\left[1-\left(\frac{r}{R^{(F)}}\right)^2\right] + \frac{3}{10}b\left[1-\left(\frac{r}{R^{F}}\right)^4\right]\right\} + \frac{S_{n0}R^{(F)2}}{3k^{(C)}}\left(1+\frac{3}{5}b\right)\left(1-\frac{R^{(F)}}{R^{(C)}}\right)$$

$$T^{(C)} = \frac{S_{n0}R^{(F)2}}{3k^{(C)}}\left(1+\frac{3}{5}b\right)\left(\frac{R^{(F)}}{r} - \frac{R^{(F)}}{R^{(C)}}\right)$$

Heat conduction with a viscous Heat Source;

Flow of incompressible Newtonian fluid b/w two coaxial cyl.





Consider the volume heat source due to viscous dissipation is S_v → not external

consider $b << R$ then $v_z = v_b\left(\frac{x}{b}\right)$ where $v_b = \Omega R$

Consider a shell of thickness Δx, width W & length L

Energy balance in the x direction,

$$W.L.e_x\Big|_x - W.L.e_x\Big|_{x+\Delta x} = 0$$

$$\frac{de_x}{dx} = 0$$

$$e_x = \left(\tfrac{1}{2}\rho v^2 + \rho H\right).v_x + \left[\tau.v\right]_x + q_x$$

x component of convective transport → 0

$[\tau.v]_x \to \tau_{xx}v_x + \tau_{xy}v_y + \tau_{xz}v_z$

first term is zero as there is no flow in the radial direction

∵ $v_x = v_y = 0$

$\tau_{xx}v_x = \tau_{xy}v_y = 0$

only $\tau_{xz}.v_z$ exists)

$e_x = C_1$

we can write

$(\tau \cdot v)_x + q_x = C_1$

$$-\mu \cdot \frac{dv_z}{dx} \cdot v_z + \left(-k \frac{dT}{dx}\right) = C_1$$

$$-\mu v_z \frac{dv_z}{dx} - k \frac{dT}{dx} = C_1 \qquad v_z = v_b\left(\frac{x}{b}\right)$$

$$-\mu\, v_b\left(\frac{x}{b}\right)\cdot\left(\frac{v_b}{b}\right) - k\frac{dT}{dx} = C_1$$

$$-k\frac{dT}{dx} - \mu x \cdot \left(\frac{v_b}{b}\right)^2 = C_1$$

$$\frac{dT}{dx} = -\frac{\mu x}{k}\left(\frac{v_b}{b}\right)^2 - \frac{C_1}{k}$$

$$T = -\frac{\mu}{k}\left(\frac{v_b}{b}\right)^2 \cdot \frac{x^2}{2} - \frac{C_1 x}{k} + C_2$$

B.C. at $x = 0$ $T = T_0$,

at $x = b$ $T = T_b$

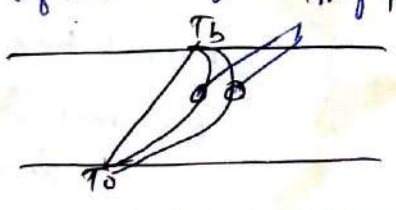
$$\boxed{\frac{T - T_0}{T_b - T_0} = \frac{1}{2}\left(\frac{\mu v_b^2}{k(T_b - T_0)}\right)\cdot \frac{x}{b}\left(1 - \frac{x}{b}\right) + \frac{x}{b}}$$

Br ⟷ Brinkman number

~~if $T_b << T_0$ then~~

~~$\frac{T - T_0}{T_0} = \frac{1}{2}\frac{\mu v_b^2}{k}$~~

For Br > 2 the max Temp. will occure at the lubricant rather than at any of the surface high temp.



(10)

Viscous heating (Sv)

= heat addition due to viscosity

$= (-\tau_{xz} \cdot W \cdot L) \cdot V_b$ } → rate of work done

$(-\tau_{xz} \cdot W \cdot L)$ → Force; V_b → $\frac{\text{displacement}}{\text{time}}$

Therefore rate of energy addition / volume

$$= -\frac{\tau_{xz}\, WL \cdot V_b}{W \cdot L \cdot b} = \mu \cdot \frac{WL\, V_b}{WLb} \cdot \left(\frac{dV_z}{dx}\right)$$

Since $V_z = \left(\frac{V_b}{b}\right)x$

$$\frac{dV_z}{dx} = \frac{V_b}{b}$$

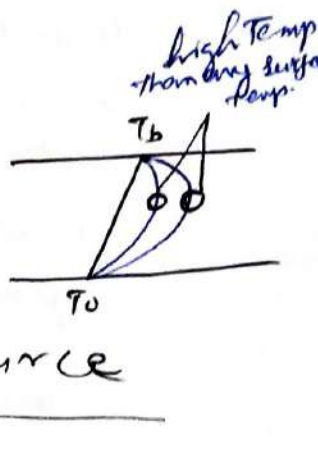
$$= \mu\left(\frac{V_b}{b}\right)^2$$

or

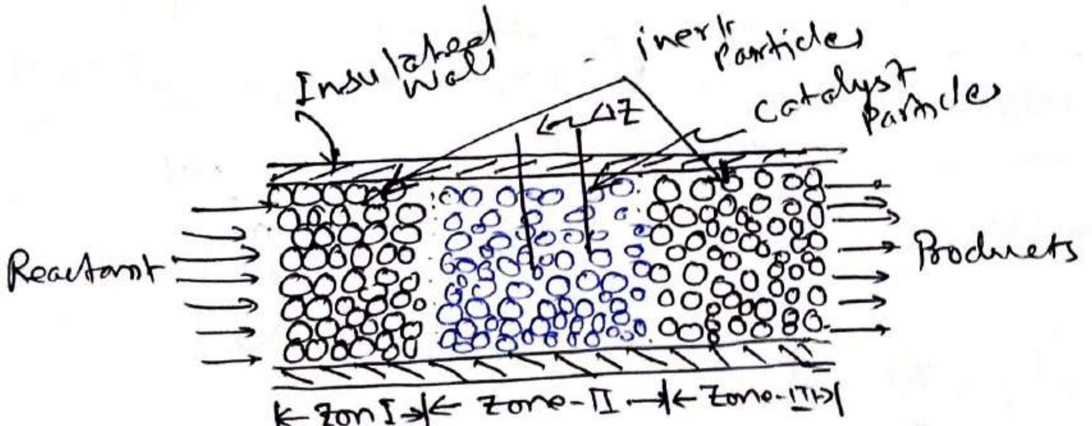
$$S_V = \mu\left(\frac{V_b}{b}\right)^2$$

(Sumanto - Lect. 35 NPTEL)

For $Br > 2$ The Lubricant will have higher Temp than any of the surface temp



Heat conduction with a Chemical Source



fixed bed axial-flow reactor. Reactants enter at $Z = -\infty$ and leave at $Z = -\infty$. The reaction zone extends from $Z = 0$ to $Z = L$

Consider that fluid is flowing in a "plug-flow" through the reactor with uniform axial velocity

$v_0 = \frac{w}{\pi R^2 \rho}$; $w = A u \rho$

& Superficial velocity

$u = v_0$ & A is x^{nal} area.

note that

(*) ρ, ρv_0 and $v_0 \neq f(r)$
$\neq f(z)$

(*) Reactor wall is insulated, so $T \neq f(r)$
but $T = f(z)$

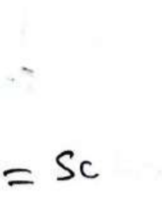
It is desired to find the temperature distribution in the z-direction when the fluid enters at $z = -\infty$ with a uniform temperature T_1.

Consider S_c be the ~~heat~~ volume rate of heat generation due to chemical reaction.

usually $S_c = f(P, T, C)$ but for simplicity let $S_c = S_1 f(\theta)$ where $\theta = \frac{T - T_0}{T_1 - T_0}$

Here T is the local temp. and in the catalyst bed and S_{c_1} & T_0 are empirical const. for the given reactor condition.

Consider a strip of Δz thickness



$$\pi R^2 e_z\big|_z - \pi R^2 e_z\big|_{z+\Delta z} + \pi R^2 \Delta z \cdot S_c = 0$$

$$\frac{de_z}{dz} = S_c$$

from $(\tau \cdot v)_z$

$$\frac{d}{dz}\left[\left(\tfrac{1}{2}\rho v^2 + \rho \hat{H}\right) v_z + \tau_{zz} v_z + q_z\right] = S_c$$

$$= \frac{d}{dz}\left(\tfrac{1}{2}\rho v_z^3 \nearrow^0 + \rho \hat{C_p}(T - T^0) v_z + (P - P^0)\nearrow^0 v_z + \rho \hat{H}_0 v_z - 2\mu v_z \frac{dv_z}{dz}\nearrow^0 - k_{eff,zz} \frac{dT}{dz}\right) = S_c$$

it may be noted that $v_x = v_y = 0$ Thus from $\tau_{zx} v_x^{\,0} + \tau_{zy} v_y^{\,0} + (\tau_{zz} v_z)$ exists.

As $v_z \neq f(z)$ also pressure gradient can be neglected
Hence the eq. can be reduced to

$$\therefore \rho \hat{C}_p (T - T_0) v_z = k_{eff,zz} \frac{dT}{dz} + S_c \quad \text{for Zone II}$$

Equations for three zones can be written as follows.

Zone I $(z < 0)$ $\rho \hat{C}_p v_0 \frac{dT^{I}}{dz} = k_{eff,zz} \frac{d^2T^{I}}{dz^2}$

Zone II $(0 < z < L)$ $\rho \hat{C}_p v_0 \frac{dT^{II}}{dz} = k_{eff,zz} \frac{d^2T^{II}}{dz^2} + S_{c_1} F(\theta)$

Zone III $(z > L)$ $\rho \hat{C}_p v_0 \frac{dT^{III}}{dz} = k_{eff,zz} \frac{d^2T^{III}}{dz^2}$

Assumption: Same $k_{eff,zz}$ for three zones

Boundary condition's for solving above BC's

(1) at $z = -\infty$ $T^{I} = T_1$

(2) at $z = 0$ $T^{I} = T^{II}$

(3) at $z = 0$ $k_{eff,zz} \frac{dT^{I}}{dz} = k_{eff,zz} \frac{dT^{II}}{dz}$

(4) at $z = L$ $T^{II} = T^{III}$

(5) at $z = L$, $k_{eff,zz} \frac{dT^{II}}{dz} = k_{eff,zz} \frac{dT^{III}}{dz}$

(6) at $z = \infty$ T^{III} = finite.

In many cases of practical interest convective heat transport is more important compared to conductive heat transport hence conductive terms could be neglected. This may be the case particularly for large Peclet number. $(Pe = Re \cdot Pr)$ i.e. at high Reynolds number ensuring plug flow behavior.

consider $\zeta = \frac{z}{L}$ &

dimensionless heat source

$$N = \frac{S_{c_1} \cdot L}{\rho \hat{C}_p v_0 (T_1 - T_0)}$$

(104) Equation for zones then reduces to

Zone. I

$(z<0)$ $\frac{d\Theta^{I}}{d\mathfrak{z}} = 0$ | As $\rho \hat{C}_p v_0 \frac{dT^{I}}{dz} = 0$

$$\frac{(T_1 - T_0)\, d\frac{(T^{I} - T_0)}{(T_1 - T_0)}}{L \cdot d(z/L)} = 0$$

or $\frac{d\Theta^{I}}{d\mathfrak{z}} = 0$

Zone - II

$0 < z < L$ $\frac{d\Theta^{II}}{d\mathfrak{z}} = N f(\theta)$

Zone III

$z > L$ $\frac{d\Theta^{III}}{d\mathfrak{z}} = 0$

We need three B.C.'s to solve above eqns

$\mathfrak{z} = -\infty$ $\Theta^{I} = 1$

$\mathfrak{z} = 0$ $\Theta^{I} = \Theta^{II}$

$\mathfrak{z} = 1$ $\Theta^{II} = \Theta^{III}$

Soln

$\Theta^{I} = 1$ Zone - I

$\int_{\Theta^{I}}^{\Theta^{II}} \frac{1}{f(\Theta)} = N\mathfrak{z}$ Zone - II

$\Theta^{III} = \Theta^{II}\Big|_{\mathfrak{z}=1}$ Zone - III

$\hookrightarrow \mathfrak{z} = \frac{z}{L} \Rightarrow \frac{L}{L} = 1$

(105)

As an approximation

$$F(\Theta) = \Theta$$

For small changes in temperature if the reaction rate is insensititive to concentration.

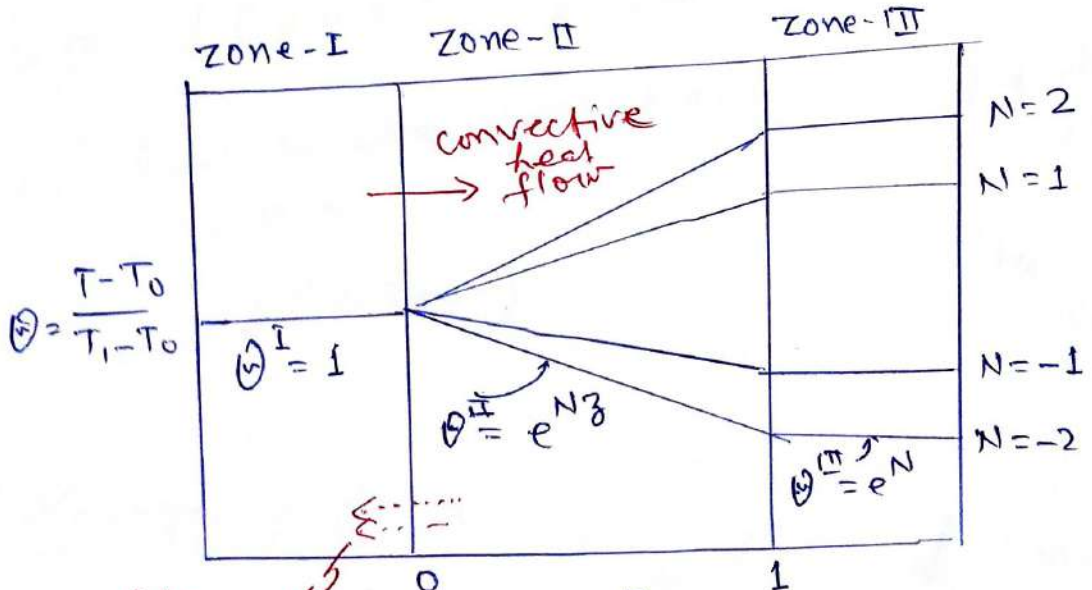
Thus we have

$$\Theta^{I} = 1 \qquad \text{Zone - I}$$

$$\Theta^{II} = e^{N\mathfrak{z}} \qquad \text{Zone - II}$$

$$\Theta^{III} = \Theta^{II}\Big|_{\mathfrak{z}=1} = e^{N(1)} = e^{N}$$

$$\therefore \Theta^{III} = e^{N} \qquad \text{Zone III}$$

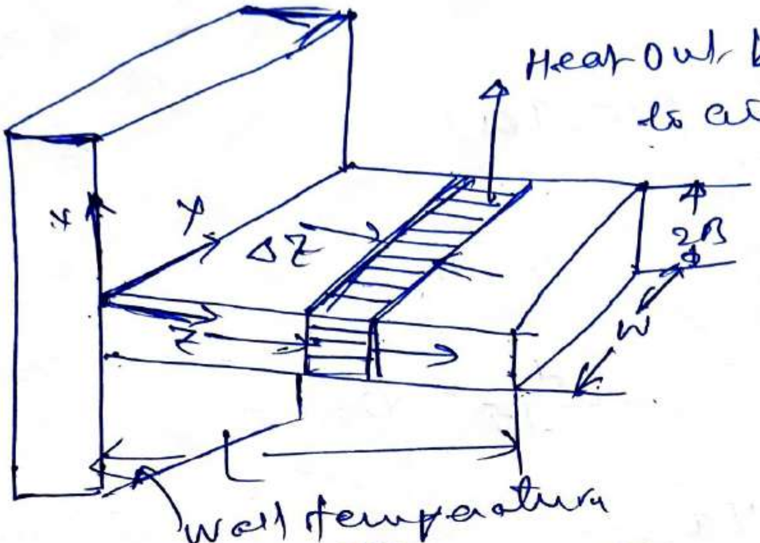


$$\mathfrak{z} = \frac{Z}{L}$$

Here in this section axial conduction has been discarded (in 10B.18, Bird it is not discarded). However in actual case when axial conduction is not discarded then both zone I & II at junction there may be pre-heating (exothermic rxn) or precooling (endothermic rxn) may occure opposit to the convective heat flow (Sunando Lect-34 NPTEL)

Heat conduction in a cooling fin

we'll find cooling fin efficiency



A single cooling fin with $B \ll L$, and $B \ll W$

$T_a \rightarrow$ ambient temp.

Actual situation

1. $T = f(x, y, z)$ — most important
2. heat is also lost from $2BW$
3. $h = f(\text{position})$

Model

$T = f(z)$

No heat loss from the edges

$q_z = h(T - T_a)$ — constant & $T = f(z)$

Energy balance:

$$2BW\, q_z|_z - 2BW\, q_z|_{z+\Delta z} - h(2W\Delta z)(T - T_a) = 0$$

Division by $2BW\Delta z$ and taking the limit as Δz approaches zero gives

$$-\frac{dq_z}{dz} = \frac{h}{B}(T - T_a)$$

$\left(q_z = -\frac{k\,dT}{dz}\right)$ in which k is the thermal conductivity of the metal.

$$\frac{d^2T}{dz^2} = \frac{h}{kB}(T - T_a)$$

B.C. 1 at $z = 0$, $T = T_w$

B.C. 2 at $z = L$ $\frac{dT}{dz} = 0$

Theta ← $\Theta = \frac{T - T_a}{T_w - T_a}$

(Zeta) ← $\zeta = \frac{z}{L}$

$N^2 = \frac{hL^2}{kB}$ dimensionless H.T.C.

$\frac{d^2\Theta}{d\zeta^2} = N^2\Theta$ with $\Theta\big|_{\zeta=0} = 1$ and

$\frac{d\Theta}{d\zeta}\Big|_{\zeta=1} = 0$

The quantity N^2 may be $N^2 = \left(\frac{hL}{k}\right)\cdot\left(\frac{L}{B}\right)$
$= Bi\left(\frac{L}{B}\right)$

Solⁿ

$\Theta = \cosh N\zeta - (\tanh N)\sinh N\zeta$

$\Theta = \frac{\cosh N(1-\zeta)}{\cosh N}$

$\eta = \frac{\text{actual rate of heat loss from the fin}}{\text{rate of heat loss from an isothermal fin at } T_w}$

(108)

η = ~~actual rate of heat loss~~

$$\eta = \frac{\int_0^W \int_0^L h\,(T - T_a)\,dz\,dy}{\int_0^W \int_0^L h\,(T_w - T_a)\,dz\,dy}$$

$$= \frac{\int_0^1 \theta\,d\zeta}{\int_0^1 d\zeta}$$

$$\eta = \frac{1}{\cosh N}\left(-\frac{1}{N}\sinh N(1-\zeta)\right)\Big|_0^1$$

$$= \frac{\tanh N}{N}$$

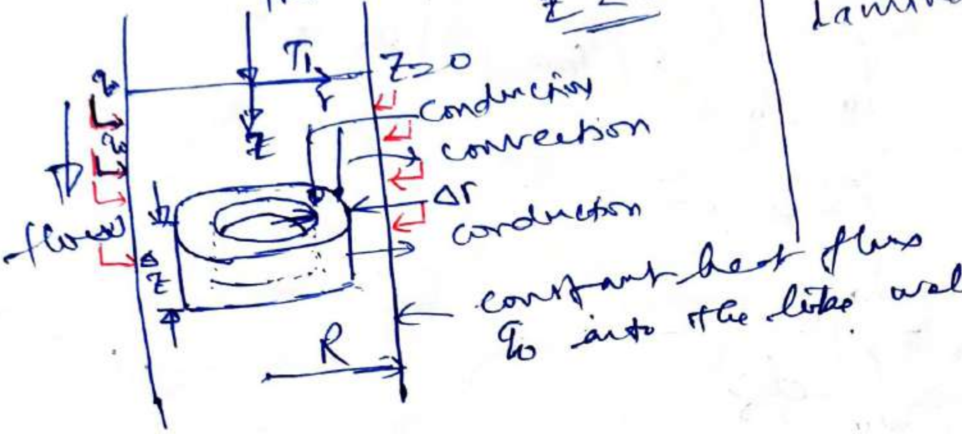
in which N is dimensionless quantity

Forced Convection

Consider forced convection in a circular tube

inlet fluid temp. for $z < 0$

Consider fluid has constant (μ, ρ, $\hat{C}_p$, k), Laminar flow



As the energy is being transported in the z and r direction consider a ring of fluid element of thickness Δr & length Δz.

(*) refer to sheet 110

Energy balance:

Total energy in at $r = e_r|_r \cdot 2\pi r \cdot \Delta z$

out at $r+\Delta r = e_r|_{r+\Delta r} \cdot 2\pi(r+\Delta r)\Delta z$

$= 2\pi r \Delta z \cdot e_r|_{r+\Delta r}$

Total energy in at $z = e_z|_z \cdot 2\pi r \cdot \Delta r$

out at $z+\Delta z = e_z|_{z+\Delta z} \cdot 2\pi r \cdot \Delta r$

work done on fluid by gravity $= \underbrace{\rho \cdot g \cdot 2\pi r \cdot \Delta r \cdot \Delta z}_{\text{force}} \cdot \underbrace{v_z}_{m/s}$

$=$ Energy/s.

In forced convection problem velocity profile is found first and then it is used to obtain the temperature profile

* Here consider the velocity profile is fully developed

$$\therefore \quad v_z = \left(\frac{\wp_0 - \wp_L}{4\mu L}\right) R^2 \left[1 - \left(\frac{r}{R}\right)^2\right]$$

$$= v_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

Energy balance

$$\frac{(r e_r)|_r - (r e_r)|_{r+\Delta r}}{\Delta r} + r \cdot \frac{e_z|_z - e_z|_{z+\Delta z}}{\Delta z} + \rho v_z g_z \cdot r = 0$$

or as $\Delta r \to 0$, $\Delta z \to 0$

$$-\frac{1}{r}\frac{\partial}{\partial r}(r \cdot e_r) - \frac{\partial e_z}{\partial z} + \rho v_z g_z = 0$$

$$e_r = q_r + \left(\tfrac{1}{2}\rho v^2 + \rho \hat{H}\right) \cancel{v_r}^{0} + (\cancel{\tau_{rr} v_r}^{0} + \cancel{\tau_{r\theta} v_\theta}^{0} + \tau_{rz} v_z)$$

$$v_z \cdot \tau_{rz} = -\mu\left(\frac{\partial v_z}{\partial r}\right) \cdot v_z$$

$$= -k\frac{\partial T}{\partial r} - \mu \frac{\partial v_z}{\partial r} \cdot v_z$$

$$e_z = q_z + \left(\frac{1}{2}\rho v^2 + \rho \hat{H}\right) v_z + (\tau_{zr} \cancel{v_r}^{0} + \tau_{z\theta} \cancel{v_\theta}^{0} + \tau_{zz} \cancel{v_z}^{0})$$

$$= -k\frac{\partial T}{\partial z} + \left(\tfrac{1}{2}\rho v^2 + \rho \hat{H}\right) v_z$$

$\hat{H}$ can be determined from law of thermodynamics

$\hat{H} = f(T, P)$

$$d\hat{H} = \left(\frac{\partial \hat{H}}{\partial T}\right)_P dT + \left(\frac{\partial \hat{H}}{\partial P}\right)_T dP = \hat{C}_p \ldots$$

$$= \hat{C}_p\, dT + \left[T\left(\frac{\partial S}{\partial P}\right)_T + V\left(\frac{\partial P}{\partial P}\right)_T\right] dP$$

Maxwell eqn $\hat{H} = T\,dS + V\,dP$

$$= \hat{C_p}\, dT + \left[T\left(-\frac{\partial \hat{V}}{\partial T}\right)_p + \hat{V} \right] dp$$

For gas consider ideal gas law follows

$$PV = RT$$

$$\therefore \left(\frac{\partial V}{\partial T}\right) = \frac{R}{P} \Rightarrow \frac{RT}{P} = V$$

$$d\hat{H} = \hat{C_p}\, dT$$

on integration

$$\hat{H} - \hat{H}^o = \hat{C_p}\,(T - T^o)$$

$\hat{C_p}$ → Assume $\neq f(T)$

For ~~liquids~~ fluid of constant density

First term will remain same

$$\hat{H} - \hat{H}^o = \hat{C_p}\,(T - T^o) + \int_{P^o}^{P} \left[\hat{V} - T\left(\frac{\partial V}{\partial T}\right)_p \right] dp$$

Fluid is incompressible means ρ is constant

$$\rho = \frac{1}{\hat{V}} \quad \text{so } \hat{V} = \text{const}$$

$$\hat{H} - \hat{H}^o = \hat{C_p}\,(T - T^o) + \int_{P^o}^{P} \left[\hat{V} - T\left(\frac{\partial \hat{V}}{\partial T}\right)_p^{\to 0} \right] dp$$

$$= \hat{C_p}\,(T - T^o) + \hat{V}\,(P - P^o)$$

$$= \hat{C_p}\,(T - T^o) + \frac{(P - P^o)}{\rho}$$

Let $\hat{H}^o \Rightarrow 0$ for reference

$$\therefore \hat{H} = \hat{C_p}\,(T - T^o) + \frac{(P - P^o)}{\rho}$$

$$e_z = -k\frac{\partial T}{\partial z} + \left(\tfrac{1}{2}\rho v^2 + \rho \hat{C}_p (T-T^0) + (p-p^0)\right) v_z$$

Substituting the terms in the shell balance eqn.

$$-\frac{1}{r}\left(\frac{\partial}{\partial r}(r e_r)\right) - \frac{\partial e_z}{\partial z} + \rho v_z g_z = 0$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\left[-k\frac{\partial T}{\partial r} - \mu\frac{\partial v_z}{\partial r}\cdot v_z\right]\right)$$

$$-\frac{\partial}{\partial z}\left[-k\frac{\partial T}{\partial z} + \frac{1}{2}\rho v_z^2 v_z + \rho\hat{C}_p(T-T^0)v_z + (p-p^0)v_z\right]$$

$$+\rho v_z g_z = 0$$

→ neglected (conduction in z direction is small)

→ 0 diff of this term will be zero as ρ const. & $v_z \neq f(z)$ ∀ z

Hence

$$\frac{1}{r}\frac{\partial}{\partial r}\left[kr\frac{\partial T}{\partial r} + \mu v_z \cdot r\frac{\partial v_z}{\partial r}\right] - \rho\hat{C}_p\frac{\partial T}{\partial z}\cdot v_z - v_z\frac{dp}{dz} + \rho v_z g_z = 0$$

$$\frac{1}{r}\left[kr\frac{\partial^2 T}{\partial r^2} + \mu v_z\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \mu r\frac{\partial v_z}{\partial r}\cdot\frac{\partial v_z}{\partial r}\right] - \rho\hat{C}_p v_z\frac{\partial T}{\partial z} - v_z\frac{\partial p}{\partial z}$$

$$+\rho v_z g_z = 0$$

$$k\frac{\partial^2 T}{\partial r^2} + \frac{k}{r}\frac{\partial T}{\partial r} + \mu\frac{v_z}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \mu\left(\frac{\partial v_z}{\partial r}\right)^2 - v_z\rho\hat{C}_p\frac{\partial T}{\partial z}$$

$$-v_z\frac{\partial p}{\partial z} + \rho v_z g_z = 0$$

→ 0 viscous heat negligible

$$k\frac{\partial^2 T}{\partial r^2} + \frac{k}{r}\frac{\partial T}{\partial r} - v_z\rho\hat{C}_p\frac{\partial T}{\partial z} + \left(\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) - \frac{\partial p}{\partial z} + \rho g_z\right)v_z = 0$$

→ 0: z-comp of velocity in N-S eqn and will be = 0 for this case. Check N-S eqn.

$$\boxed{\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = v_z\rho\hat{C}_p\frac{\partial T}{\partial z}} \quad \text{—— (1)}$$

Desired expression.

Now v_z is a fn of r as:

$$v_z = v_{z\,max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

(113) Thus

$$\rho \hat{C_p} V_{z\,max}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial z} = \frac{k}{r}\left[\frac{\partial}{\partial r}\left(r\cdot\frac{\partial T}{\partial r}\right)\right] \quad \text{—(2)}$$

To solve this eqn an alternate method is given in Arihant heat transfer PP-259

Boundary conditions

(1) at $r=0$; $\frac{dT}{dr}=0$ symmetry $\forall z$
or
$T=$ finite

(2) at $r=R$ $\quad -k\frac{\partial T}{\partial r}=q_0$ uniform heat flux at wall
∴ continuity of heat flux.

(3) at $z=0$ $\quad T=T_1 \quad \forall r$

Soln of above eqn involves the use of dimensionless parameters

$$\Theta = \frac{T-T_1}{\frac{q_0 R}{k}} \;;\quad \xi=\frac{r}{R} \;;\quad \zeta = \frac{z}{\rho \hat{C_p} v_{z,max}\frac{R^2}{k}}$$

(Theta) (xi) (Zeta)

$\zeta = \frac{z}{R}\cdot\frac{\mu}{D V_{z\,max}\,\rho}\cdot\frac{k}{\mu C_p}$

$\zeta = \frac{z}{R}\cdot\frac{1}{Re}\cdot\frac{1}{Pr}$.

Re & Pr are important in Forced Convection correlation

The choice for dimension less temp is from (2) & (3) B.C.

The eqn then Becomes

$$(1-\xi^2)\frac{\partial\Theta}{\partial\zeta} = \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Theta}{\partial\xi}\right) \text{ with B.C.} \quad \text{—(3)}$$

at $\xi=0$; $\Theta=$ finite ; at $\xi=1$, $\frac{\partial\Theta}{\partial\xi}=1$; at $\zeta=0$, $\Theta=0$

An asymptotic solution for the above eqn could be obtained for large ζ. As for large ζ the temperature profile as a function of ξ will not undergo further change with increasing ζ. Thus for large ζ

$$\Theta(\xi,\zeta) = C_0\zeta + \psi(\xi) \quad \text{—(4)}$$

where C_0 is a constant to be determined.

However this eqn does not satisfy B.C. 3, but satisfies B.C. 1 and B.C. 2. Hence B.C. 3 needs to be changed.

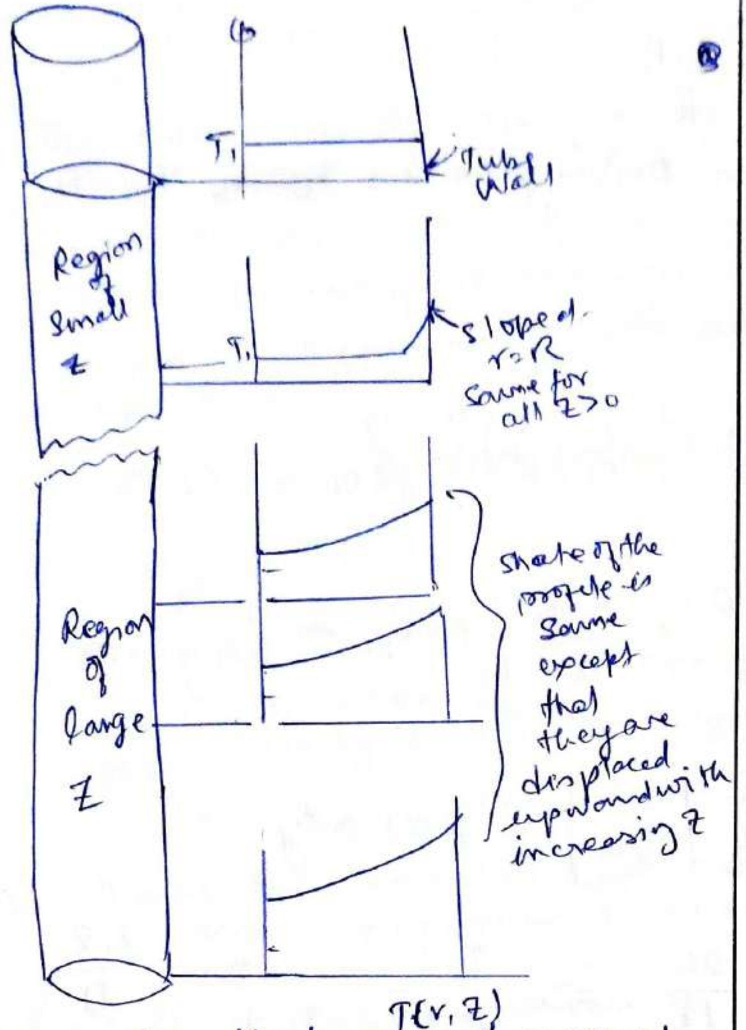


Figure: how the temperature would change when the tube wall is heating using a coil wrapped around the tube uniformly

B.C. 4:

$$2\pi R z q_0 = \int_0^{2\pi}\int_0^R \rho \hat{C}_p (T - T_1) v_z \, r\,dr\,d\theta$$

or in dimensionless form.

$$\zeta = \int_0^1 \theta(\xi, \zeta)(1-\xi^2)\xi \, d\xi \quad — (5)$$

i.e. Energy supplied over a distance ζ is = the (energy leaving at ζ − energy entering at $\zeta = 0$)

Substituting eqn (4) into eqn (3)

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi \frac{d\psi}{d\xi}\right) = C_0(1-\xi^2)$$

which gives on twice integration.

$$\theta(\xi, \zeta) = C_0\zeta + C_0\left(\frac{\xi^2}{4} - \frac{\xi^4}{16}\right) + C_1 \ln \xi + C_2$$

Using B.C.s (1), (2) and (4)

the constants are $C_1 = 0$ from B.C. 1

$C_0 = 4$ — B.C. 2

$C_2 = -7/24$ from condition 4

Thus

$$\Theta = 4\zeta + \xi^2 - \frac{1}{4}\xi^4 - \frac{7}{24}$$

Valid for large ζ; $\zeta \to \infty$

Arithmetic avg temp.

$$\langle T \rangle = \frac{\int_0^{2\pi}\int_0^R T(r,z)\, r\,dr\,d\theta}{\int_0^{2\pi}\int_0^R r\,dr\,d\theta} = T_1 + \left(4\zeta + \frac{7}{24}\right)\frac{q_0 R}{k}$$

Bulk avg. temp or mixing cup temp.

$$T_b = \frac{\langle v_z T \rangle}{\langle v_z \rangle} = \frac{\int_0^{2\pi}\int_0^R v_z(r)\, T(r,z)\, r\,dr\,d\theta}{\int_0^{2\pi}\int_0^R v_z(r)\, r\,dr\,d\theta}$$

$$T_b = T_1 + 4\xi \frac{q_0 R}{K}$$

Local Heat Transfer Driving Force, $T_W - T_b$

@ $r = R_1$, $T = T_W$

$$\therefore \frac{T - T_1}{\frac{q_0 R}{K}} = 4\xi + \left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4 - \frac{7}{24}$$

$$\underbrace{T_W - T_1 = 4\xi \frac{q_0 R}{K}}_{\to T_b} + \frac{q_0 R}{K}\left[1 - \frac{1}{4} - \frac{7}{24}\right]$$

$$T_W - T_b = \frac{q_0 R}{K}\left[\frac{11}{24}\right] = f(r) \text{ only.}$$

or

$$\frac{q_0}{K(T_W - T_b)} \cdot R = \frac{24}{11} \Rightarrow \frac{q_0}{K(T_W - T_b)} \cdot D = \frac{48}{11}$$

$$\because q_0 = h(T_W - T_b)$$

$$\boxed{\frac{hD}{K} = \frac{48}{11}} \Rightarrow \text{limiting value of Nusselt Number}$$

The Nusselt number depends upon Re & Pr in case of Forced convection.

Refer to Page No. 235-247 Geankoplis.

Heat Transfer – (i) Fourier's Law
(ii) Notes on Th. Conductivity
(iii) Derivations – Parallel wall, Composite wall, cylindrical wall
and Numericals based on them.

The equation of change for Nonisothermal system.

Law of conservation of energy. which is an extension of first law of thermodynamics will be applied over a differential volume to obtain the energy equation.

First law of thermodynamics

$\Delta U = Q + W$

Q involves : entering and leaving K.E & I.E. due to conv. & cond.

W → { work done by Body forces line, gravity
Surface forces such as, pressure, viscous force. }

Thus the general expression for the energy conservation thus becomes

Rate of increase of K.E. & I.E. = Net rate of K.E. & I.E. addition by conv. transport + Net rate of heat addition by molecular transport conduction + rate of work done on system by molecular mechanism i.e. by stresses (P, τ etc.) + rate of work done due to external forces. e.g. by gravity ——— (1)

Mathematically

$$L.H.S. = \Delta x \cdot \Delta y \cdot \Delta z \cdot \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho \hat{U}\right)$$

$\rho\hat{U}$ → I.E./vol.

$\hat{U} \rightarrow \frac{\text{Energy}}{\text{mass}}$

$\frac{1}{2}\rho v^2 = \frac{1}{2}\rho(v_x^2 + v_y^2 + v_z^2)$ ↳ K.E/vol.

The first three terms of the R.H.S of the eqn are present in $\underline{e}$ i.e. combined energy flux vector.

Energy entering the volume element $\Delta x \Delta y \Delta z$

(117)

$$= \Delta y \Delta z (e_x|_x - e_x|_{x+\Delta x}) + \Delta z \Delta x (e_y|_y - e_y|_{y+\Delta y}) + \Delta x \Delta y (e_z|_z - e_z|_{z+\Delta z}) \quad \text{--- (2)}$$

work done on fluid due to gravity force (external forces)

$$= \rho \Delta x \Delta y \Delta z (g \cdot v) = \rho \Delta x \Delta y \Delta z (g_x v_x + g_y v_y + g_z v_z) \quad \text{--- (3)}$$

From (1), (2) & (3)

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2 + \rho \hat{U}\right) = -\left(\frac{\partial e_x}{\partial x} + \frac{\partial e_y}{\partial y} + \frac{\partial e_z}{\partial z}\right) + \rho (g_x v_x + g_y v_y + g_z v_z)$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2 + \rho \hat{U}\right) = -(\nabla \cdot e) + \rho (v \cdot g)$$

Extending vectore

$$e = \left(\tfrac{1}{2}\rho v^2 + \rho \hat{U}\right) v + q + p v + [\tau \cdot v]$$

Thus

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2 + \rho \hat{U}\right) = \left(\nabla \cdot \left(\tfrac{1}{2}\rho v^2 + \rho \hat{U}\right) v\right) - (\nabla \cdot q) - (\nabla \cdot p v) - (\nabla \cdot [\tau \cdot v]) + \rho (v \cdot g)$$

Terms	Description
L.H.S.	rate of Increase of K.E. & I.E. per unit vol.
R.H.S. (1) ——	rate of energy addition due to Conv. transport per vol.
(2) ——	conductive transport (Molecular transport)
(3) —	work done by pressure force / vol.
(4) —	work done by viscous force. τ has same unit as p
(5) —	work done by gravity force.

$p \rightarrow$ Force / Area | $\frac{\text{Force} \times \frac{m}{s}}{\text{Area}} \times \frac{1}{m}$

$v \rightarrow m/s$

$\nabla \rightarrow 1/m$ | $\frac{\text{Energy rate}}{\text{Vol.}}$

Above eqn doesn't include nuclear, radioactive, electromagnetic or chemical forms of energy.

Special Forms of Energy eqn:

The energy eqn is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\hat{U}\right) = -\left(\nabla\cdot\left(\frac{1}{2}\rho v^2 + \rho\hat{U}\right)\bar{v}\right) - (\nabla\cdot\bar{q}) - (\nabla\cdot p\bar{v}) - \left(\nabla\cdot[\bar{\tau}\cdot\bar{v}]\right) + \rho(\bar{v}\cdot g)$$

From this we subtract the Mechanical energy eqn

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\left(\nabla\cdot\frac{1}{2}\rho v^2\bar{v}\right) - (\nabla\cdot p\bar{v}) - p(-\bar{\nabla}\cdot\bar{v}) - (\bar{\nabla}\cdot[\bar{\tau}\cdot\bar{v}]) - (-\bar{\tau}:\bar{\nabla}\bar{v}) + \rho(\bar{v}\cdot\bar{g})$$

$$\frac{\partial}{\partial t}(\rho\hat{U}) = -(\bar{\nabla}\cdot\rho\hat{U})\bar{v} - (\nabla\cdot\bar{q}) + p(-\bar{\nabla}\cdot\bar{v}) + (-\bar{\tau}:\bar{\nabla}\bar{v})$$

irreversible rate of I.E. increase / vol by viscous Dissipation

or ~~For inviscous ... fluid~~

$$\rho\frac{D\hat{U}}{Dt} = -(\bar{\nabla}\cdot\bar{q}) - p(\bar{\nabla}\cdot\bar{v}) - (\bar{\tau}:\bar{\nabla}\bar{v})$$

→ More explanation in Appendix A.4

Reversible rate of internal energy increase per unit volume by compression.

$$\tau:\nabla v = \tau_{xx}\frac{\partial v_x}{\partial x} + \tau_{xy}\frac{\partial v_x}{\partial y} + \tau_{xz}\frac{\partial v_x}{\partial z} + \tau_{yx}\frac{\partial v_y}{\partial x} + \tau_{yy}\frac{\partial v_y}{\partial y} + \tau_{yz}\frac{\partial v_y}{\partial z} + \tau_{zx}\frac{\partial v_z}{\partial x} + \tau_{zy}\frac{\partial v_z}{\partial y} + \tau_{zz}\frac{\partial v_z}{\partial z}$$

Further

$$\hat{U} = \hat{H} - pV = \hat{H} - (p/\rho)$$

$\therefore$

$$\frac{D\hat{U}}{Dt} = \frac{D\hat{H}}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt}$$

$\therefore$

$$\rho\frac{D\hat{H}}{Dt} = -(\bar{\nabla}\cdot\bar{q}) - (\bar{\tau}:\bar{\nabla}\bar{v}) + \frac{Dp}{Dt} \quad \text{--- (1)}$$

$$\rho\frac{D\hat{H}}{Dt} = \rho\hat{C}_p\frac{DT}{Dt} + \rho\left[\hat{V} - T\left(\frac{\partial\hat{V}}{\partial T}\right)_p\right]\frac{Dp}{Dt} \quad \text{from eqn 9.8-7}$$

$$= \rho\hat{C}_p\frac{DT}{Dt} + \rho\left[\frac{1}{\rho} - T\left(\frac{\partial 1/\rho}{\partial T}\right)_p\right]\frac{Dp}{Dt}$$

$$\rho \frac{D\hat{H}}{Dt} = \rho \hat{C}_p \frac{DT}{Dt} + \left[1 + \left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P\right] \frac{DP}{Dt} \quad -(2)$$

Substituting this value into eqn (1) we have

$$\boxed{\rho \hat{C}_p \frac{DT}{Dt} = -(\bar{\nabla} \cdot \bar{q}) - (\bar{\tau} : \bar{\nabla} \bar{v}) - \left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P \frac{DP}{Dt}}$$

eqn of chang for temperature

When fourier's law is used $-(\nabla \cdot q) = (\nabla \cdot K \nabla T)$

if K is constant then $= (K \cdot \nabla^2 T)$

Special Cases

(i) For ideal gas $\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_P = -1$ So,

$$\rho \hat{C}_p \frac{DT}{Dt} = K \nabla^2 T + \frac{DP}{Dt}$$

(ii) For fluid flowing in a constant pressure system

$$\frac{DP}{Dt} = 0 \;\therefore$$

$$\boxed{\rho \hat{C}_p \frac{DT}{Dt} = K \nabla^2 T}$$

(iii) For fluid with constant density

$$\left(\partial \ln \rho / \partial \ln T\right)_P = 0$$

$$\therefore \boxed{\rho \hat{C}_p \frac{DT}{Dt} = K \nabla^2 T}$$

(iv) For a stationary solid, v is zero hence

$$\boxed{\rho \hat{C}_p \frac{\partial T}{\partial t} = K \nabla^2 T} \rightarrow \text{Fourier's eqn}$$

Relevant dimensionless groups:

$$Re = \left[\frac{l_0 v_0 \rho}{\mu}\right] = \text{Reynolds number}$$

$$Pr = \left[\frac{c_p \mu}{k}\right] = \frac{\nu}{\alpha} = \text{Prandtl number}$$

$$Gr = \left[g\beta (T_1 - T_0) l_0^3 / \nu^2\right] = \text{Grashof Number}$$

$$Br = \left[\mu v_0^2 / k (T_1 - T_0)\right] = \text{Brinkman Number}$$

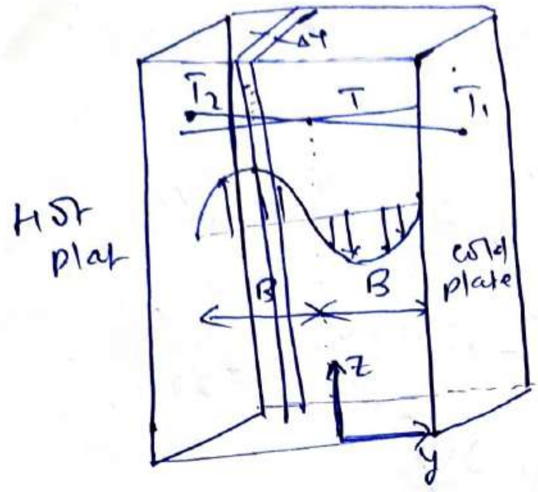
$$Pe = Re\,Pr = \text{Peclet number}$$

$$Ra = Gr\,Pr = \text{Rayleigh Number}$$

$$Ec = Br/Pr = \text{Eckert Number}$$

Refer to Table 11.5-3

Free convection Problem:



ex. Flow pattern b/w two parallel plates maintained at different temperatures

Fluid of density ρ and viscosity μ is located b/w the plates

It is assumed that temp difference is sufficiently small.

* System is closed at the top & bottom.

Due to the temp diff. the fluid at hot end rises and that on cold end descends and the velocity profile as shown develops

* The plates are assumed to be very tall so that end effects can be neglected.
* Temperature is a fn. of 'y' alone.

Select a shell a thickness of Δy to make energy balance.

in 'y' direction there is no convection and heat transfer is only by conduction (neglect the viscous heating term)

$$\therefore \; -\frac{dq_y}{dy} = 0 \quad \text{or} \quad k\frac{d^2T}{dy^2} = 0$$

at $y = -B, \; T = T_2$, $y = +B \; T = T_1$

$$\therefore \boxed{T = \bar{T} - \tfrac{1}{2}\Delta T \, y/B}$$

$\bar{T} = \frac{1}{2}(T_1 + T_2)$

$\Delta T = T_2 - T_1$

Now let's find velocity distribution

make shell balance over the ~~th~~ Δy slab.

$\phi_{xz}, \phi_{yz}, \phi_{zz}$

$v_x = v_y = 0$

$$\phi_{xz} = \rho v_x v_z + p\,0 + \left(-\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right) = 0$$

$$\phi_{yz} = \rho v_y v_z + p\,0 + \left[-\mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)\right] = -$$

$$\phi_{zz} = \rho v_z^2 + p + \left[-2\mu \frac{\partial v_z}{\partial z}\right]$$

on making balance

$$\mu \frac{d^2 v_z}{dy^2} = \frac{dp}{dz} + \rho g$$

$\mu \rightarrow$ assumed constant

$\rho = f(T)$ ∵ Natural convection

As the ΔT is small change in ρ will be small hence ρ can be expanded about $\bar{T}$ using Taylor series

$$\therefore \rho = \rho\Big|_{T=\bar{T}} + \frac{d\rho}{dT}\Big|_{T=\bar{T}} (T-\bar{T}) + \cdots$$

$$= \bar{\rho} - \bar{\rho}\bar{\beta}(T-\bar{T})$$

$\beta \rightarrow$ volume expansion coefficient

$$\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = \frac{1}{(1/\rho)}\left(\frac{\partial (1/\rho)}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$

∴

$$\mu \frac{d^2 v_z}{dy^2} = \frac{dp}{dz} + \left[\bar{\rho} - \bar{\rho}\bar{\beta}(T-\bar{T})\right] g$$

~~$= \frac{dp}{dz} + \left[\bar{\rho} - \bar{\rho}\bar{\beta}\ \tfrac{y}{2}\ \Delta T (y/B)\right] g$~~

Note that the Temperature change is small. Hence the density change will be small.
Assume that at $\bar{T} = \frac{1}{2}(T_2 + T_1)$

$$\rho = \bar{\rho}$$

Using Taylor series expansion ρ can be then expanded around $\bar{T}$ as

$$\rho = \bar{\rho}\Big|_{T=\bar{T}} + \frac{d\rho}{dT}\Big|_{T=\bar{T}} (T - \bar{T})$$

$(\rho, T) \quad (\bar{\rho}, \bar{T})$

$$= \bar{\rho} - \bar{\rho}\bar{\beta}(T - \bar{T})$$

$\bar{\rho}$, & $\bar{\beta}$ are the density and the volume expansion coefficient at $\bar{T}$

β is defined as

$$\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = \frac{1}{(1/\rho)}\left(\frac{\partial (1/\rho)}{\partial T}\right)_p$$

$$= -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$

Therefore

$$\mu \frac{d^2 v_z}{dy^2} = \frac{dp}{dz} + \left(\bar{\rho} - \bar{\rho}\bar{\beta}(T - \bar{T})\right) \cdot g$$

$$\mu \frac{d^2 v_z}{dy^2} = \left(\frac{dp}{dz} + \bar{\rho} g\right) - \bar{\rho} g \bar{\beta}(T - \bar{T})$$

viscous force — pressure force — gravity force — Buoyancy force.

But $T = \bar{T} - \frac{1}{2}\Delta T \frac{y}{B}$ ∴

$$\mu \frac{d^2 v_z}{dy^2} = \left(\frac{dP}{dz} + \bar{\rho} g\right) - \bar{\rho} g \bar{\beta} \left(\bar{T} - \tfrac{1}{2}\Delta T \tfrac{y}{B} - \bar{T}\right)$$

$$= \left(\frac{dP}{dz} + \bar{\rho} g\right) + \tfrac{1}{2}\bar{\rho} g \bar{\beta} \Delta T \, (y/B)$$

B.C.'s

(1) at $y = -B$, $v_z = 0$,

(2) at $y = +B$ $v_z = 0$

~~$v_z = \frac{(\bar{\rho} g \bar{\beta} \Delta T) B^2}{12\mu} \left[(y/B)^3 - (y/B) \right] + B^2$~~

$$\mu \frac{dv_z}{dy} = \left(\frac{dP}{dz} + \bar{\rho} g\right) \cdot y + \tfrac{1}{4} \bar{\rho} g \bar{\beta} \left(\Delta T \cdot y^2/B\right) + C_1$$

$$\mu v_z = \left(\frac{dP}{dz} + \bar{\rho} g\right) \cdot \frac{y^2}{2} + \frac{1}{12} \bar{\rho} g \bar{\beta} \left(\Delta T \frac{y^3}{B}\right) + C_1 y + C_2$$

from (1) B.C.

$$0 = \left(\frac{dP}{dz} + \bar{\rho} g\right) \cdot \frac{B^2}{2} - \frac{1}{12} \bar{\rho} g \bar{\beta} . \Delta T . B^2 - C_1 B + C_2 \quad \text{—— (X)}$$

II B.C

$$0 = \left(\frac{dP}{dz} + \bar{\rho} g\right) \cdot \frac{B^2}{2} + \frac{1}{12} \bar{\rho} g \bar{\beta} \left(\Delta T B^2\right) + C_1 B + C_2 \quad \text{—— (Y)}$$

(X + Y)

$$C_2 = -\left(\frac{dP}{dz} + \bar{\rho} g\right) \cdot \frac{B^2}{2}$$

$\therefore$ from X

$$C_1 = -\frac{1}{12} \bar{\rho} g \bar{\beta} \Delta T . B$$

~~$\frac{dP}{dz} + \bar{\rho} g \frac{B^2}{2}$~~ $c_1 = -\frac{1}{12} \rho g \bar{\beta} \Delta T \underline{B}$ (P2)

$$\therefore v_z = \frac{1}{2\mu}\left(\frac{dP}{dz} + \bar{\rho} g\right)(y^2 - B^2)$$

$$+ \frac{1}{12\mu} \bar{\rho} g \bar{\beta} \Delta T \frac{y^3}{B} - \frac{1}{12} \bar{\rho} g \bar{\beta} \Delta T B . y$$

$$v_z = \frac{B^2}{2\mu}\left(\frac{dP}{dz} + \bar{\rho} g\right)\left(\left(\frac{y}{B}\right)^2 - 1\right)$$

$$+ \frac{1}{12\mu} \bar{\rho} g \bar{\beta} \Delta T \, y . B \left(\frac{y^2}{B^2} - 1\right)$$

~~$\frac{B^2}{B}\left(\frac{y}{B}\right)\left(\frac{y^2}{B^2} - 1\right)$~~

~~$B^2\left(\left(\frac{y^3}{B}\right) - \left(\frac{y}{B}\right)\right)$~~

$$\therefore v_z = \frac{1}{12\mu} \bar{\rho} g \bar{\beta} \Delta T B^2 \left[\left(\frac{y}{B}\right)^3 - \left(\frac{y}{B}\right)\right] + \frac{B^2}{2\mu}\left(\frac{dP}{dz} + \bar{\rho} g\right)\left[\left(\frac{y}{B}\right)^2 + \ldots\right.$$

Mass Balance

The net mass flow in the z direction is zero

$$\int_{-B}^{+B} \rho v_z \, dy = 0 \qquad \frac{dP}{dz} = -\bar{\rho} g$$

substitute

$$\rho = \bar{\rho} - \bar{\rho}\bar{\beta}\left(\frac{1}{2}\Delta T \, y/B\right)$$

& v_z from above eqn

Just remember it is (±)B & −B in the ~~even terms~~ limits so the ~~...~~ after integration will even power of y after integration will cancel out and only odd power term in second term will remain which yields v_z expression. And that yields

$\left(\frac{dP}{dz} + \bar{\rho} g\right) \cdot$ coefficient = 0

$$\therefore \boxed{\frac{dP}{dz} + \bar{\rho} g = 0}$$

Therefor the expression for v_3 becomes.

$$v_3 = \frac{(\bar{\rho} g \bar{\beta} \Delta T) B^2}{12 \mu} \left[\left(\frac{y}{B}\right)^3 - \left(\frac{y}{B}\right) \right]$$

avg velocity of upward moving stream

$$\langle v_3 \rangle = \frac{\int_{-B}^{0} v_3 \, dy \cdot W}{(-B \cdot W)} = \frac{\frac{W \bar{\rho} g \bar{\beta} \Delta T B^2}{12 \mu} \left[\frac{y^4}{4B^3} - \frac{y^2}{2B} \right]_{-B}^{0}}{(-B \cdot W)}$$

$$= \frac{\frac{\bar{\rho} g \bar{\beta} \Delta T B^2}{48 \mu} [B] \cdot W}{(B \cdot W)} = \frac{1}{48} \frac{\bar{\rho} g \bar{\beta} \Delta T B^2}{\mu}$$

These expression for v_3 shows that fluid motion is a consequence of buoyant force associated with the temperature gradient.

Let's define a dimensionless velocity

$$V_3 = \frac{B v_3 \bar{\rho}}{\mu} \quad \& \quad Y = (y/B)$$

thus

$$V_3 = \frac{1}{12} Gr (Y^3 - Y)$$

where Grashoff number = Gr

$$= \left[\frac{(\bar{\rho}^2 g \bar{\beta} \Delta T) B^3}{\mu^2} \right]$$

$$Gr = \frac{\bar{\rho} g B^3 \bar{\rho} \bar{\beta}}{\mu^2} (T_2 - T_1) = \frac{\bar{\rho} g B^3}{\mu^2} \left[\bar{\rho} \bar{\beta} \left[(T_2 - \bar{T}) - (T_1 - \bar{T}) \right] \right]$$

$$= \frac{\bar{\rho} g B^3}{\mu^2} \left[\bar{\rho} \bar{\beta} \Delta T_2 - \bar{\rho} \bar{\beta} \Delta T_1 \right]$$

$$= \frac{\bar{\rho} g B^3}{\mu^2} \left[\underbrace{\bar{\rho} - \bar{\rho} \bar{\beta} \Delta T_1}_{\rho_1} - \underbrace{(\bar{\rho} - \bar{\rho} \bar{\beta} \Delta T_2)}_{\rho_2} \right]$$

$$Gr = \frac{\bar{\rho} g B^3}{\mu^2} \Delta \rho$$

$\Delta \rho = \rho_1 - \rho_2$
Note
$\Delta T = T_2 - T_1$

~~UNSTEADY STATE HEAT TRANSFER in~~ solids

For solids the governing heat transfer eq^n.

$$\rho \hat{C}_p \frac{\partial T}{\partial t} = \nabla \cdot q$$

$q = k \nabla T$

$$= \nabla \cdot k \nabla T$$

$k \rightarrow$ isotropic

assumed $k \neq f(T)$

$$= k \nabla^2 T$$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p} \nabla^2 T \Rightarrow \frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

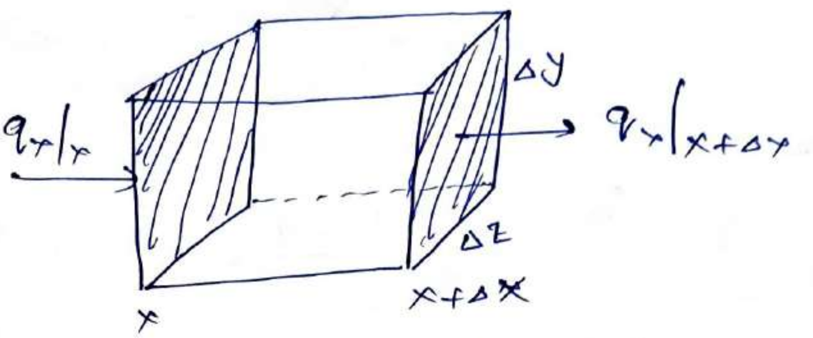
$\frac{k}{\rho \hat{C}_p} \rightarrow \alpha$

~~Heating a Semi-infinite slab~~

Unsteady-state heat transfer

Basic eqn

Consider a cube of dimension Δx, Δy, & Δz,



$q_x = -kA\frac{\partial T}{\partial x}$ for heat conduction in x direct.

Heat Balance

rate of heat (input - output) + generation = Acc.

$$\left(q_x|_x - q_x|_{x+\Delta x}\right) + \dot{q}\,(\Delta x.\Delta y.\Delta z) = \Delta x.\Delta y.\Delta z.\rho\, C_p \frac{\partial T}{\partial t}$$

$\dot{q}$ → rate of heat generation/vol.

$$\therefore \dot{q} - \frac{\partial q_x}{\partial x} = \rho C_p \frac{\partial T}{\partial t}$$

if $\dot{q} = 0$ then

~~$\rho C_p \frac{\partial T}{\partial t} = -\frac{\partial q_x}{\partial x} = \partial T\, kA$~~

$$\rho C_p \frac{\partial T}{\partial t} =$$

$$\left(k\frac{\partial T}{\partial x}\Big|_x - k\frac{\partial T}{\partial x}\Big|_{x+\Delta x}\right).\Delta y.\Delta z + \Delta x\,\Delta y\,\Delta z\,\dot{q} = \Delta x\,\Delta y.\Delta z.\rho C_p \frac{\partial T}{\partial t}$$

$$\dot{q} + k\frac{\partial^2 T}{\partial x^2} = \rho c_p \frac{\partial T}{\partial t}$$

or

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_p}$$

$$= \alpha \frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_p}$$

k, ρ, c_p are assumed constant

SI Units: $\alpha \to m^2/s$, $T \to K$, $t \to s$, $k = W/m.K$
$\rho \to kg/m^3$, $\dot{q} = W/m^3$, $c_p = J/kg.K$

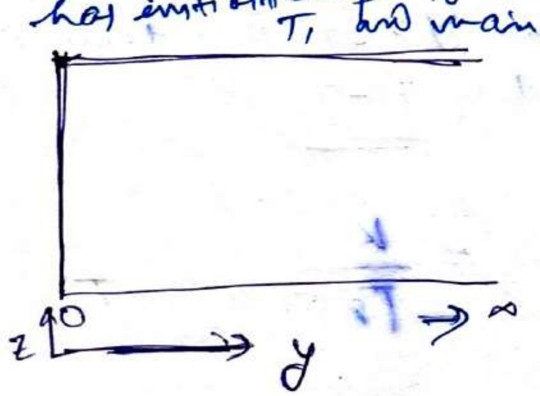
For three dimension case

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{\dot{q}}{\rho c_p}$$

$$\boxed{\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{\dot{q}}{\rho c_p}}$$

Heating a semi-infinite slab

A solid material occupying the space from $y=0$, to $y=\infty$ has initial temp T_0 and at $t=0$ temp at $y=0$ is raised to T_1 and maintained at it. Obtain the temp profile



at $t<0$ $T_{y>0} = T_0$ I.C

$t\geq 0$ ∀ >0 $T_{y=0} = T_1$ B.C

Find $T(y,t)$

Sol^n define $\theta = \frac{T-T_0}{T_1-T_0}$

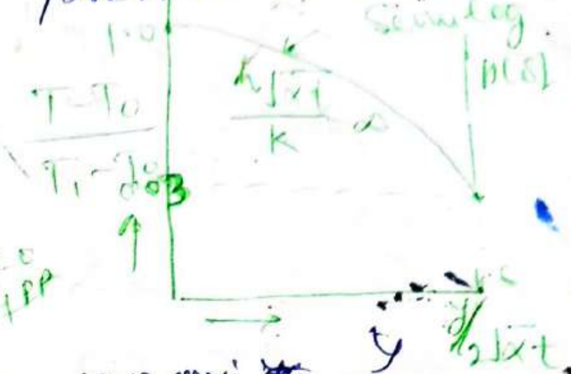
I.C. $t \leq 0$ $\theta = 0$ $\forall y$

B.C. 1 $y=0$ $\theta = 1$ $\forall t>0$

$y=\infty$ $\theta = 0$ $\forall t>0$

$$\theta = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{y}{\sqrt{4\alpha t}}} \exp(-\eta^2)\, d\eta$$

[Look into previous momentum eqn]



$$\therefore \frac{T - T_0}{T_1 - T_0} = 1 - \text{erf} \frac{y}{\sqrt{4\alpha t}}$$

Plot $\frac{T - T_0}{T_1 - T_0}$ on x axis $\frac{y}{\sqrt{4\alpha t}}$

when $\frac{y}{\sqrt{4\alpha t}} = 2$ Then $\text{erf} \frac{y}{\sqrt{4\alpha t}} = 0.99$

$$\frac{T - T_0}{T_1 - T_0} = 0.01$$

$$\therefore y = 4\sqrt{\alpha t}$$

$$\delta_T = 4\sqrt{\alpha t}$$ thermal penetration thickness.

That means for distance $y > \delta_T$ the temperature has change by less than 1% of $T_1 - T_0$

Wall heat flux

$$q_y \big|_{y=0} = -k \frac{\partial T}{\partial y}\bigg|_{y=0} = \frac{k}{\sqrt{\pi \alpha t}} (T_1 - T_0)$$

$$q_y \big|_{y=0} \propto t^{-1/2},$$
&
$$\delta_T \propto t^{1/2}$$

Read Pg no. 330 - 336 geankoplis

Biot number and Nusselt number are similar but the difference is in conductivity in Nu it is liquid conductivity whereas in Bi it is solid conductivity. Further, Bi number tells us whether Lumped capacitance model is used? if Bi is less than 0.2 then lumped capcitance model is used to get the temperature of an object.
In lumped capacitance model the spatial variation of the temperature withing the object is neglected.

UNIT - IV

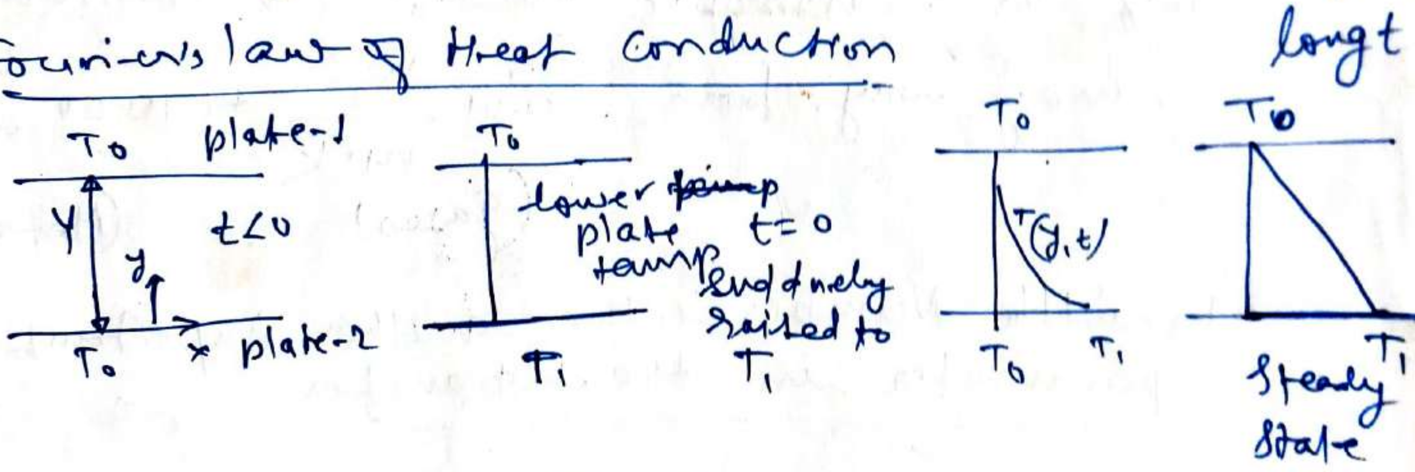
Energy Transport Phenomena

Energy Transport
- → Conduction (Molecular energy transport position of molecules are kind of fixed)
- → Convection (Bulk motion for the fluid)
  - (i) In convection at molecular level the transfer is still through conduction.
  - (ii) In convection molecules changes its position.
- → Diffusive transport → inter diffusing mixture
- → Radiation → No medium is reqd.

Thermal conductivity describes at what rate heat is conducted in a material

~~H.T. in fluid~~

Fourier's law of Heat Conduction



To maintain $\Delta T = T_1 - T_0$
certain amount of heat must be supplied say (Q)

$$\frac{Q}{A} = -k\frac{\Delta T}{y}$$ or in differential form

$$\boxed{q_y = -k\frac{dT}{dy}}$$

Heat flow ⊕ive quantity
Flow in ⊕ive y direction

For temperature variation in three dimensional form.

$$q_x = -K_x \frac{\partial T}{\partial x} \quad \text{—— (1)}$$

$$q_y = -k_y \frac{\partial T}{\partial y} \quad \text{—— (2)}$$

$$q_z = -k_z \frac{\partial T}{\partial z} \quad \text{—— (3)}$$

$$\bar{q} = \delta_x \cdot q_x + \delta_y \cdot q_y + \delta_z \cdot q_z$$

$$\boxed{\bar{q} = -K \nabla T}$$ Three dimentional form of Fourier's law.

It describes molecular transport of heat by in a isotropic media (k constant)

k - may vary from $0.01 \frac{W}{m \cdot k}$ (gases) to $1000 \frac{W}{m \cdot k}$ (Metals)

Prandtl Number is the another important parameter in Heat transfer

Pr number for gases ⟶ low 0.x

liquids ⟶ x.0 to xxxx.0

glycerol 30°C – 6580

350°C – 329

Energy is a Scalar
Momentum is a vector

(56)

Pr → liquid metals — very low 0.00XX to 0.0XX
(Na, K, → 0.00XX; Hg, Pb, Bi → 0.0XX)

(See liquid metals handbook)
with T → may be high/low (see 9.1-4 Bird)

k of solids → with temp. k may be high or low

Ex Al. 373.2 → 205.9
573.2 → 268   873.2 → 423

Cd 273.2 → 93.0
373.2 → 90.4

wood — parallel to axis → 0.126; normal to axis → 0.038 | anisotropic

Temperature & pressure dependence of Thermal conductivity:-

~~you may refer~~ to the monograph in fig 9.2-1
* for gases k↑ with T↑, for liquid k↓ with T↑

Chapman enskog formula for k of a monoatomic gas of low density & T

$$k = 1.9891\times10^{-4}\frac{\sqrt{T/M}}{\sigma^2\Omega_k}$$

$k \to \frac{cal}{cm.s.K}$

$T \to K$

$\sigma \to A^\circ$ (collision diameter)

$\Omega_k \to$ collision integral for thermal conductivity of Lennard-Jones potential Table E.2

REFER Sect 9.5 Bird

by comparing with or similar viscosity formula $\mu = 2.6693\times10^{-5}\frac{\sqrt{MT}}{\sigma^2\Omega_\mu}$

eqn is good for monoatomic gas say Ne, Xe, Ar, etc.

for polyatomic gas at low density

$$k = \left(\hat{C}_p + \frac{5}{4}\frac{R}{M}\right)\mu$$ Eucken's formula (Oy-ken's)

as a special case $\hat{C}_p = 5/2\ R/M$ for monoatomic gas.

$$\left(k = \frac{15}{4}\frac{R}{M}\mu = \frac{5}{2}\hat{C}_v\mu\right)$$ for monoatomic gas

^ → circumflex for per/mass
~ → Tilde for per mole

For mix. of gas at low density

$$k_{mix} = \sum_{\alpha=1}^{N} \frac{x_\alpha k_\alpha}{\sum_\beta x_\beta \phi_{\alpha\beta}}$$

$x_\alpha \rightarrow$ mole fraction
$k_\alpha \rightarrow$ The cond. of pure gas
$\phi_{\alpha\beta} \rightarrow$ constant

Q. 9.3-1

Compute the thermal conductivity of a monoatomic gas of low density

For Ne → parameter (Leonard-Jones) from Table E.1

$\sigma = 2.789\ A^\circ$, $\epsilon/k = 35.7\ K$, $M = 20.183$ unit. wt.

(ϵ = characteristic energy)

∴ at 373 K $kT/\epsilon = \frac{373.2}{35.7} = 10.45$

from Table E.2 $\Omega_k = \Omega_\mu = 0.821$

Now $k = 1.981 \times 10^{-4} \frac{\sqrt{T/M}}{\sigma^2 \Omega_k}$

$= 1.981 \times 10^{-4} \frac{\sqrt{373.2/20.1}}{(2.789)^2 (0.821)}$

$= 1.338 \times 10^{-4}\ \frac{cal}{cm \cdot s \cdot K}$

Measured value $= 1.37 \times 10^{-4}\ \frac{cal}{s \cdot cm \cdot K}$

9.3-2

Estimate the thermal conductivity of molecular oxygen at 300K and low pressure

(The conductivity of Polyatomic gases at low density)

Mol. wt. of O_2 = 32.0 $\tilde{C}_{p,300K} = 7.019 \frac{cal}{g.mol.K}$

from Table E.1 Leonard Jones parameter for molecular oxygen to be

$\sigma = 3.433 A^\circ$ and $\epsilon/\kappa = 113 K$

At 300K then $\kappa T/\epsilon = \frac{300}{113} = 2.655$

Table E.2 $\Omega_\mu = 1.074$ The viscosity

from Eq. 1.4-18

$$\mu = 2.6693\times10^{-5} \frac{\sqrt{MT}}{\sigma^2 \Omega_\mu}$$

$$= 2.6693\times10^{-5} \frac{\sqrt{32.00 \times 300}}{(3.433)^2(1.074)}$$

$$= 2.065\times10^{-4} \text{ g/cm.s}$$

from Eucken approximation

$$k = \left(\tilde{C}_p + \frac{5}{4}R\right)\left(\mu/M\right)$$

$$= (7.079 + 2.484)(2.065\times10^{-4})/32.00$$

$$= 6.14\times10^{-5} \frac{cal}{cm.s.K}$$

9.3-3 do your self.

* Thermal conductivity of liquids

$$k = 2.80 \left(\frac{\tilde{N}}{\tilde{V}}\right)^{2/3} K v_s$$ (modified Bridgman's formula)

$\left(\frac{\tilde{V}}{\tilde{N}}\right) \rightarrow$ volume / molecule

modification 3 → 2.80

eqn applicable to low densities well above critical density

- Sonic velocity → v_s

$\frac{\tilde{V}}{\tilde{N}} \rightarrow$ volume / molecule

$K \rightarrow$ Boltzmann constant

The velocity of low frequency sound

$$v_s = \sqrt{\frac{C_p}{C_v}\left(\frac{\partial p}{\partial \rho}\right)_T}$$

$\left(\frac{\partial p}{\partial \rho}\right)_T \rightarrow$ may be obtained from eqn of state

$\left(\frac{C_p}{C_v}\right) \rightarrow 1$ for liquids except near critical point

Prediction of the thermal conductivity of a liquid

9.4-1 ~~Do it yourself~~. The density of liquid CCl_4 at 20°C and 1 atm is 1.595 g/cm³, and its isothermal compressibility $\frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T = 90.7 \times 10^{-6}$ atm⁻¹. What is the thermal conductivity?

Soln $\left(\frac{\partial p}{\partial \rho}\right)_T = \frac{1}{\rho\left(\frac{1}{\rho}\right)\left(\frac{\partial \rho}{\partial p}\right)_T} = \frac{1}{1.595 \times 90.7 \times 10^{-6}} = 6.91 \times 10^{3}\ \frac{\text{atm cm}^3}{g}$

$6.91 \times 10^{3} \times 1.0133 \times 10^{6}\ \frac{g}{cm\,s^2} \cdot \frac{cm^3}{g} = 7.00 \times 10^{9}\ \frac{cm^2}{s^2}$

Assuming $\frac{C_p}{C_v} = 1.0$ (for liquids) $\rightarrow v_s = \sqrt{\frac{C_p}{C_v}\left(\frac{\partial p}{\partial \rho}\right)_T} = 8.37 \times 10^{4}\ \frac{cm}{s}$

molar volume $\tilde{V} = \frac{M}{\rho} = \frac{153.84}{1.595} = 96.5\ \frac{cm^3}{g\,mole}$

from $k = 2.80\left(\frac{N}{V}\right)^{2/3} K v_s = 2.80\left(\frac{6.023 \times 10^{23}}{8.37 \times 10^{21}}\right)^{2/3} \frac{(1.3815 \times 10^{-16})}{(8.37 \times 10^{4})}$

$= 1.10 \times 10^{4} (cm^{-2}/(\frac{ergs}{k})(cm/s))$

$= 0.110 \frac{W}{m \cdot K}$

Thermal conductivity of solids:

Th.c.(solid) should be measured experimentally as they depend on many factors (porosity, orientation, fluid contained in the pores)

Heat conductione. ~~Electricity conductione~~

~~Pure metal~~ ~~good~~

Pure metals better heat conductors than nonmetals. Crystalline metals conduct heat more readily than amorphous materials. Dry porous poor heat conductors. ~~and are therefore~~

	Temp ↑	The cond. ↑
Metals	↑	↓ ——— ↑
Non metals	——— ↑	
Alloys	↑	intermediate behaviours

rel^n with electrical thermal conductivity for pure metal (Lorenz eq^n)

$$\frac{k}{k_e T} = L$$

$k_e \rightarrow$ electrical Th. C.

$L \rightarrow$ Lorenz number

$\simeq 22 - 29 \times 10^{-9} \frac{volt^2}{k^2}$

for pure metals at 0°C

"L" increases by 10–20% per 1000°C Typically.

At very low temp metals becomes super conductor of electricity hence L varies rapidly in the at low temp. region. (Superconducting region)

Assignments 9A1–9A5
9A8, 9A10

Effective Thermal conductivity of solid

Solids with pores or solid dispersed in ~~other~~ another solid (Two phase solid).

It can be treated as a homogeneous material of thermal conductivity (k_{eff})

* Convective Transport of Energy:

Transport due to bulk motion of fluid across the surface element ds ⊥ to the x axis is

$$= \underbrace{\left(\tfrac{1}{2}\rho v^2 + \rho\hat{U}\right)}_{\text{Energy}/\text{vol.}} \underbrace{v_x\, ds}_{\text{vol. flow}/s} \rightarrow \frac{\text{Energy}}{s}.$$

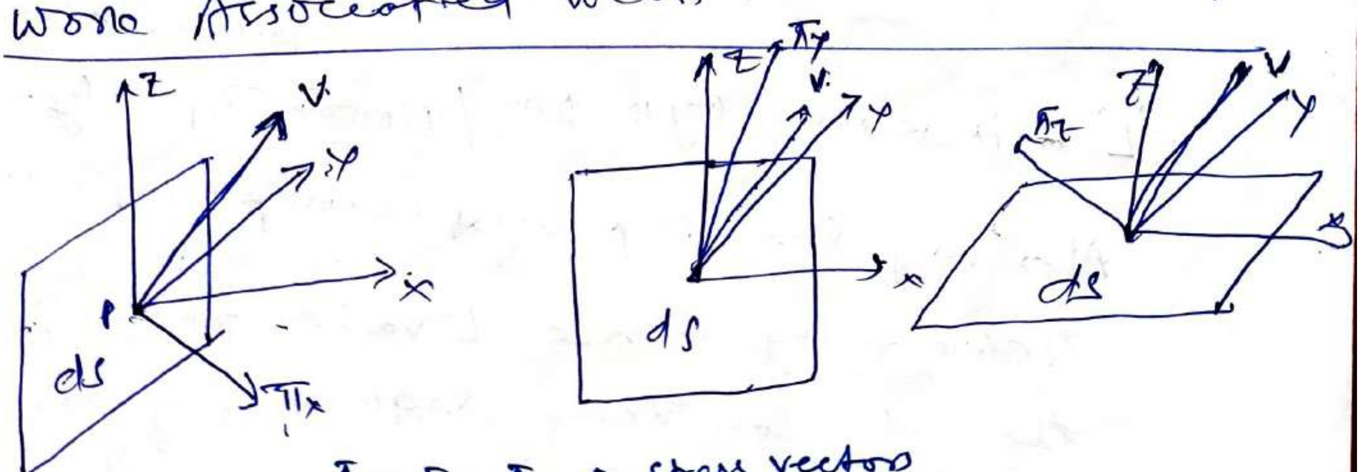
Note $\frac{1}{2}\rho v^2 = \frac{1}{2}\rho(v_x^2+v_y^2+v_z^2)$

convective energy flux $= \left(\tfrac{1}{2}\rho v^2+\rho\hat{U}\right)\delta_x v_x + \left(\tfrac{1}{2}\rho v^2+\rho\hat{U}\right)\delta_y v_y + \left(\tfrac{1}{2}\rho v^2+\rho\hat{U}\right)\delta_z v_z$

$$= \left(\tfrac{1}{2}\rho v^2+\rho\hat{U}\right)\mathbf{v}$$

It is called convective energy flux vector. It is a flux from ⊖ ive side to ⊕ ive side

Work Associated with Molecular motion:



$\pi_x, \pi_y, \pi_z \rightarrow$ stress vectors

71.a

As the fluid is moving with velocity v

rate of work done by (−)ve fluid on (+) fluid is

Note: The fluid on the minus side of the surface exerts a force $\pi_x ds$ on the fluid that is on the (+)ive side.

$(\pi_x \cdot v) ds$

in the component form

$(\pi_x \cdot v) = \pi_{xx} v_x + \pi_{xy} v_y + \pi_{xz} v_z \equiv [\pi \cdot v]_x$

$(\pi_y \cdot v) = \pi_{yx} v_x + \pi_{yy} v_y + \pi_{yz} v_z \equiv [\pi \cdot v]_y$

$\pi \to$ Molecular Stress Tensor

$\pi = p\delta + \tau$

molecular work flux vector

$\delta_x(\pi_x \cdot v) + \delta_y(\pi_y \cdot v) + \delta_z(\pi_z \cdot v)$

so that $[\pi \cdot v] = pv + [\tau \cdot v]$

The term pv can then be combined with the internal energy term $\rho U v$ to give an enthalpy term

$\rho \hat{U} v + p v = \rho [\hat{U} + p/\rho] v$

$= \rho(\hat{U} + p\hat{V}) v = \rho \hat{H} v$

$e = (\frac{1}{2}\rho v^2 + \rho \hat{U}) v + [\pi \cdot v] + q$

or $e = (\frac{1}{2}\rho v^2 + \rho \hat{H}) v + [\tau \cdot v] + q$ → molecular heat flux vector

convective energy flux

e → Combined energy flux vector.

Enthalpy can be represented as a fn of temp by

$\hat{H} - \hat{H}_0 = \int_{T^0}^{T} \hat{C}_p \, dT + \int_{p^0}^{p} \left[\hat{V} - T\left(\frac{\partial \hat{V}}{\partial T}\right)_p \right] dp$

enthalpy/mass at the reference state.

integral over p $= 0$ (for ideal gas)

$= \frac{1}{\rho}(p - p^0)$ for fluids of an const ρ

(91.b)

The integral over T becomes

$C_p^{\circ}(T - T^{\circ})$ if the heat capacity can be regarded as const. over the relevant temp. range.

Assignment 9A.1, 9A.2, 9A.3, 9A.4,
9A.5, 9A.8, ✓9A.10

Chapter 10 Bird.

Shell Energy Balance & Temp distribution in solids and Laminar flow.

General Energy Balance eqn at S.S.

rate of (~~Heat~~ (in − out) conven + (in − out) mol. transport + ~~(transport)~~)

(work done on syst by − work done by system) mol. transport + work done by External system Energy + ~~transport~~] = 0

This is a I law of thermo written for open system.

Above eqn generates a ~~I order~~ ODE for this slab to be solved with suitable B.C.

Common B.C

① Specify the surface temp.

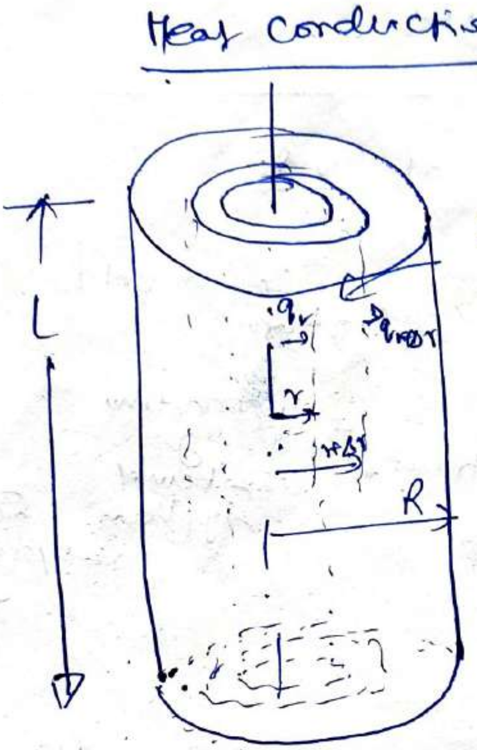
② heat flux normal to a surface may be given (as good as specifying the normal comp of the temp. gradient ~~for temp.~~

③ ~~Temp continuity at the surface and heat flux~~ Temp & heat flux continuity at the surface

④ at solid fluid surface

$q = h(T_0 - T_b)$ Newton's law of cooling

Heat conduction with an Electrical Source:



Uniform heat production by electrical heating S_e.

wire of Radius R and electrical conductivity k_e ohm⁻¹ cm⁻¹

let rate of heat production

$$\boxed{S_e} = \frac{I^2}{k_e}$$

heat source due to electrical dissipation

Assuming that the temp rise is not so large

then k & $k_e \neq f(T)$

The surface of the wire is maintained at T_0

Q.2 Find the radial temp distribution

for energy balance consider a shell of thickness Δr and length L.

Since $v = 0$, eqn is

$e = \sum (\rho v^2 + \rho \hat{H}) v + (\tau \cdot v) + q$

$\therefore e = q$

$q_r \rightarrow$ heat flux molecular

$(q_r - q_{r+\Delta r}) 2\pi r L$ $\quad \Delta r \rightarrow$ very small

&

Rate of heat production = $(2\pi r \Delta r L) . S_e$.

combine

$$(r q_r - r q_{r+\Delta r}) 2\pi L + 2\pi r \Delta r L . S_e = 0$$

$$-\frac{d(q r)}{dr} + S_e r = 0$$

$$\frac{d(q r)}{dr} = S_e r \quad \text{or}$$

~~..., q_r is ...~~

$$q r = \frac{S_e r^2}{2} + C_1$$

or

$$q = \frac{S_e r}{2} + \frac{C_1}{r}$$

at $r = 0$, q is finite $\therefore C_1 = 0$

$$\boxed{q_r = \left(\frac{S_e r}{2}\right)}$$

$$q_r = -k \frac{dT}{dr}$$

$$\therefore -k \frac{dT}{dr} = \frac{S_e r}{2}$$

at $r = R$, $T = T_0$ — B.C.

$$\therefore \boxed{T - T_0 = \frac{S_e R^2}{4k}\left[1 - \left(\frac{r}{R}\right)^2\right]}$$

It is a parabolic fn.

(95)

(i) Max temp rise

$$T_{max} - T_0 = \frac{S_e R^2}{4k}$$

(ii) Avg temp rise

$$\langle T \rangle - T_0 = \frac{\int_0^{2\pi}\int_0^R (T(r) - T_0)\, r\,dr\,d\theta}{\int_0^{2\pi}\int_0^R r\,dr\,d\theta} = \frac{S_e R^2}{8k}$$

$$= \frac{1}{2} T_{max}$$

(iii) Heat outflow at the surface (for a length L of wire)

$$Q\big|_{r=R} = 2\pi R L \cdot q_{r=R}$$
$$= 2\pi R L \cdot \frac{S_e R}{2} = S_e \cdot \pi R^2 L$$

clearly heat input = heat output at S.S

Comparison with Momentum transport

	Tube flow	Heated wire
First integration	$\tau_{rz}(r)$	$q_r(r)$
IInd integration	$v_z(r)$	$T(r) - T_0$
B.C.s $r=0$	τ_{rz} = finite	q_r = finite
$r=R$	$v_z = 0$	$T - T_0 = 0$
Property (Transport)	μ	k
Source term.	$\left(\frac{P_0 - P_L}{L}\right)$	S_e
Assumption.	μ = const	k, k_e = const

Geankoplis 4th ed. - check

Thermal conductivity of gas.

$k \propto (\sqrt{T})$ (Chapman eqn)

$K \neq f(P)$ but at very low pressure

$k \to 0$

Th. cond. of liquid: refer (Reid et al. 1977)

Energy is transferred due to mol. collision.

* Reid et al. → properties of gases and liquids
Prausnitz

$k = a + bT$ $k \neq f(P)$

$k_{water} > k_{organic\ liqd.}$

Th. cond. of solid: varies quite widely

metals have very high Th. c.

wood, rock etc. have very low Th. c.

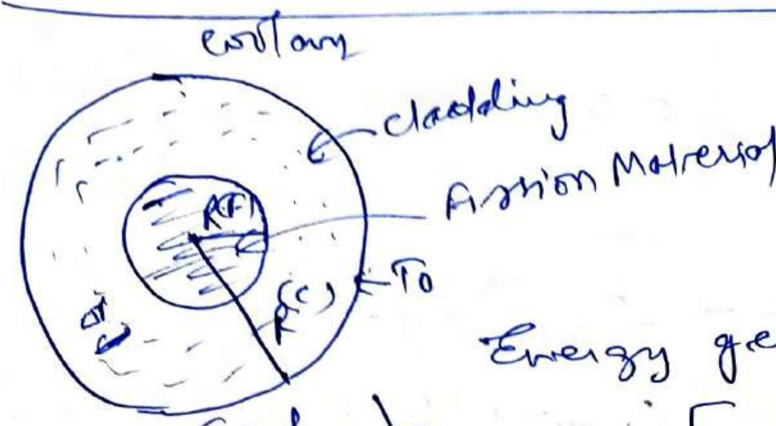
Two mechanism of H.T in solid

(1) heat is conducted by free electron

(2) Heat is conducted by translational energy.

Assignment | 10.2.1 example | 10.2.2

Heat conduction with a Nuclear heat Source :-



Energy generated from nuclear fission ~~$S_n = S_{n0}\left[1 + b\left(\frac{r}{R^{(F)}}\right)^2\right]$~~

$\left(\frac{cal}{cm^3 \cdot s}\right) \leftarrow S_n = S_{n0}\left[1 + \frac{b}{1}\left(\frac{r}{R^{(F)}}\right)^2\right]$

$S_{n0} \rightarrow$ volume rate of heat production at the centre of the sphere, and b is a dimensionless (+)ive constant.

, No flow hence $e = q$

consider a sphere of thickness Δr.

rate of heat in: $q_r^{(F)}\big|_r \cdot 4\pi r^2$,

rate of heat Out: $q_r^{F}\big|_{r+\Delta r} \; 4\pi(r+\Delta r)^2 \simeq 4\pi r^2 q_r^{(F)}\big|_{r+\Delta r}$

Rate of thermal energy produced by nuclear fission: $S_n \cdot 4\pi r^2 \Delta r$

making a balance

$$\frac{d}{dr}\left(r^2 q_r^{(F)}\right) = S_n r^2 \qquad \Delta r \to 0$$

(97)

$$\frac{d}{dr}\left(r^2 q_r^{(F)}\right) = S_{n0}\left[1 + b\left(\frac{r}{R^{(F)}}\right)^2\right] r^2$$

For cladding

$$\frac{d}{dr}\left(r^2 q_r^{(C)}\right) = 0$$

on integration

$$q_r^{(F)} = S_{n0}\left[\frac{r}{3} + \frac{b}{R^{(F)2}} \cdot \frac{r^3}{5}\right) + \frac{C_1^{(F)}}{r^2}$$

$$q_r^{(C)} = \frac{C_1^{(C)}}{r^2}$$

from II nd B.C. at $r = R^{(F)}$ $q_r^{(F)} = q_r^{(C)}$

$$\therefore q_r^{(F)}\Big|_{R^{(F)}} = \frac{C_1^{(C)}}{R^{(F)2}}$$

$$S_{n0}\left[\frac{R^{(F)}}{3} + \frac{b}{R^{(F)2}} \cdot \frac{R^{(F)3}}{5}\right) = \frac{C_1^{(C)}}{R^{(F)2}}$$

$$\therefore C_1^{(C)} = S_{n0}\left[\frac{1}{3} + \frac{b}{5}\right] R^{(F)3}$$

B.C.s

at $r = 0$, $q_r^{(F)}$ is finite

at $r = R^{(F)}$ $q_r^{(F)} = q_r^{(C)}$ continuity of flux

$$q_r^{(F)} = S_{n0}\left(\frac{r}{3} + \frac{b}{R^{(F)2}} \cdot \frac{r^3}{5}\right)$$ Flux distribution in fissionable material

$$q_r^{(C)} = S_{n0}\left(\frac{1}{3} + \frac{b}{5}\right)\frac{R^{(F)3}}{r^2}$$ cladding

~~Substitute~~ Substitute Fourier's law to find temp distribution

$$-k^{(F)}\frac{dT^{(F)}}{dr} = S_{n0}\left(\frac{r}{3} + \frac{b}{R^{(F)2}} \cdot \frac{r^3}{5}\right)$$

$$-k^{(C)}\frac{dT^{(C)}}{dr} = S_{n0}\left(\frac{1}{3} + \frac{b}{5}\right)\frac{R^{(F)3}}{r^2}$$

$$T^{(F)} = -\frac{S_{n0}}{k^{(F)}}\left(\frac{r^2}{6} + \frac{b}{R^{(F)2}}\frac{r^4}{20}\right) + C_3^{(F)}$$

$$T^{(C)} = \frac{S_{n0}}{k^{(C)}}\left(\frac{1}{3} + \frac{b}{5}\right)\frac{R^{(F)3}}{r} + C_3^{(C)}$$

B.C.s

at $r = R^{(F)}$ $T^{(F)} = T^{(C)}$ continuity of temp

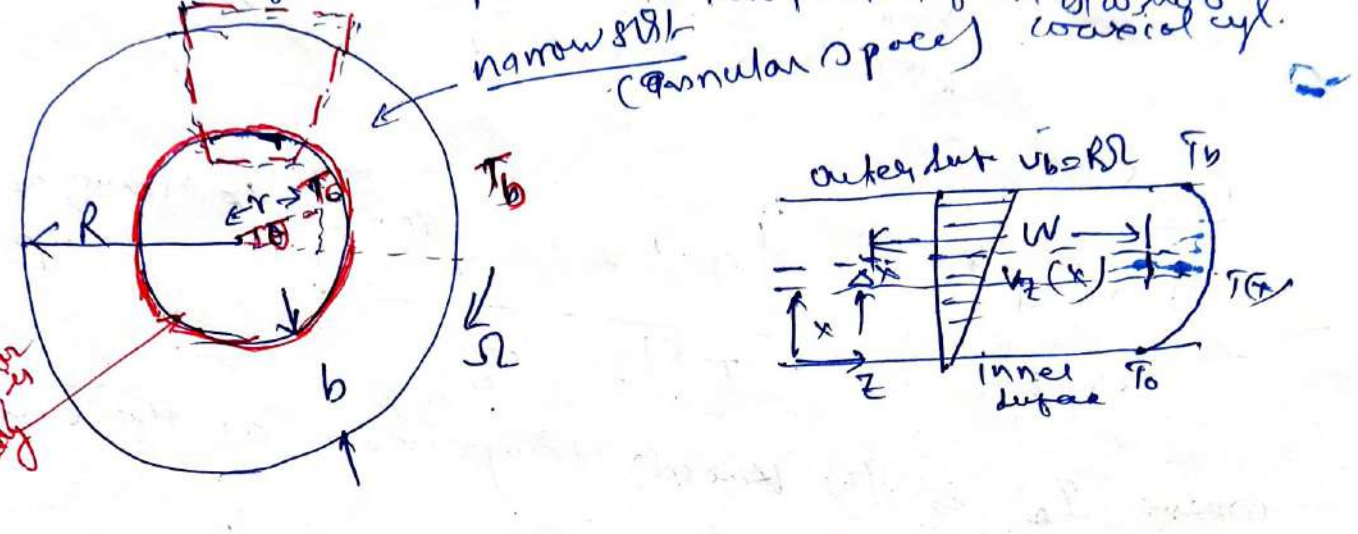
$r = R^{(C)}$ $T^{(C)} = T_0$

where T_0 is the known temperature at the outside of the cladding

$$T^{(F)} = \frac{S_{n0} R^{(F)2}}{6k^{(F)}}\left\{\left[1 - \left(\frac{r}{R^{(F)}}\right)^2\right] + \frac{3}{10}b\left[1 - \left(\frac{r}{R^{F}}\right)^4\right]\right\} + \frac{S_{n0} R^{(F)2}}{3k^{(C)}}\left(1 + \frac{3}{5}b\right)\left(1 - \frac{R^{(F)}}{R^{(C)}}\right)$$

$$T^{(C)} = \frac{S_{n0} R^{(F)2}}{3k^{(C)}}\left(1 + \frac{3}{5}b\right)\left(\frac{R^{(F)}}{r} - \frac{R^{(F)}}{R^{(C)}}\right)$$

Heat conduction with a Viscous Heat Source;
Flow of incompressible Newtonian fluid b/w two coaxial cyl.



Consider the volume heat source due to viscous dissipation is S_v → not external

consider $b << R$ then $v_z = v_b\left(\frac{x}{b}\right)$
where $v_b = \Omega R$

Consider a shell of thickness Δx, width W & length L

Energy balance in the x direction,

$$W.L.e_x\Big|_x - W.L.e_x\Big|_{x+\Delta x} = 0$$

$$\frac{de_x}{dx} = 0$$

$$e_x = \left(\tfrac{1}{2}\rho v^2 + \rho H\right).v_x + [\tau.v]_x + q_x$$

($[\tau.v]_x \to \tau_{xx}v_x + \tau_{xy}v_y + \tau_{xz}v_z$; first term → 0: x component of convective transport; $\tau_{xx}v_x$, $\tau_{xy}v_y$ → 0)

first term is zero as there is no flow in the radial direction

∵ $v_x = v_y = 0$
$\tau_{xx}v_x = \tau_{xy}v_y = 0$
only $\tau_{xz}.v_z$ exists)

~~$-\rho\frac{dv_z}{dx}\tau_{xz} + q_x$~~

$e_x = C_1$

we can write

$(\tau \cdot v)_x + q_x = C_1$

$$-\mu \frac{dv_z}{dx} \cdot v_z + \left(-k \frac{dT}{dx}\right) = C_1$$

$$-\mu v_z \frac{dv_z}{dx} - k \frac{dT}{dx} = C_1 \qquad v_z = v_b\left(\frac{x}{b}\right)$$

$$-\mu v_b\left(\frac{x}{b}\right)\cdot\left(\frac{v_b}{b}\right) - k\frac{dT}{dx} = C_1$$

$$-k\frac{dT}{dx} - \mu x \cdot \left(\frac{v_b}{b}\right)^2 = C_1$$

$$\frac{dT}{dx} = \frac{-\mu x}{k}\left(\frac{v_b}{b}\right)^2 - \frac{C_1}{k}$$

$$T = -\frac{\mu}{k}\left(\frac{v_b}{b}\right)^2 \cdot \frac{x^2}{2} - \frac{C_1 x}{k} + C_2$$

B.C. at $x = 0$ $T = T_0$,

at $x = b$ $T = T_b$

$$\boxed{\frac{T - T_0}{T_b - T_0} = \frac{1}{2}\left(\frac{\mu v_b^2}{k(T_b - T_0)}\right) \cdot \frac{x}{b}\left(1 - \frac{x}{b}\right) + \frac{x}{b}}$$

$\frac{\mu v_b^2}{k(T_b - T_0)}$ → Brinkman number.

~~if $T_b << T_0$~~ then

~~$\frac{T - T_0}{T_0} = \frac{1}{2} \frac{\mu v_b^2}{k}$~~

Viscous heating (S_v)

heat addition due to viscosity

$= \left(-\tau_{xz} \, WL \right) \cdot v_b$ ↳ force ↳ velocity } rate of work done

* Plane ⊥ to x direction momentum in z direction.

∴ Rate of energy addition / volume

$$-\tau_{xz} \frac{W \cdot L \cdot v_b}{WL \cdot b} = \mu \frac{dv_z}{dx} \cdot \left(\frac{v_b}{b}\right)$$

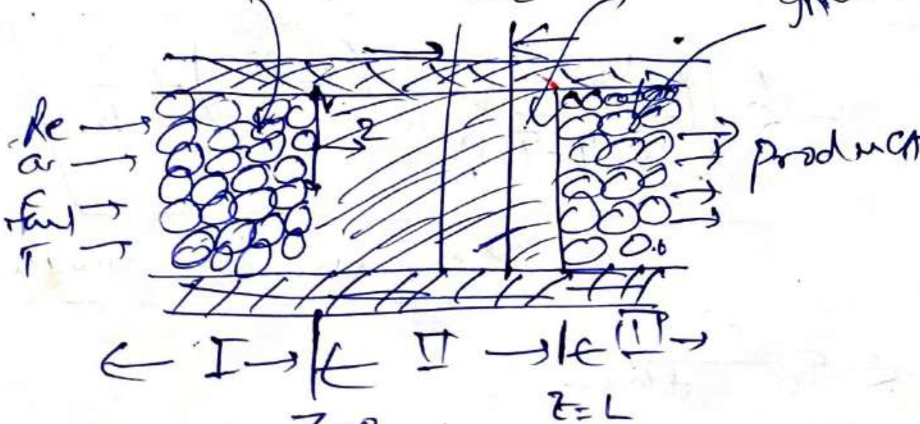
$$v_z = v_b \left(\frac{x}{b}\right)$$

$$\frac{dv_z}{dx} = \left(\frac{v_b}{b}\right)$$

$$= \mu \left(\frac{v_b}{b}\right)^2$$

$$S_v = \mu \left(\frac{v_b}{b}\right)^2$$

Heat conduction with a Chemical Source



Fixed bed axial flow reactor.
Reactant enter at $z = -\infty$ and leave at $z = \infty$

Consider the fluid is flowing in a plug flow manner with axial uniform velocity $v_0 = \frac{w}{\pi R^2 \rho}$

$w = A u \rho$

$u = v_0$

$A \to$ X-nal Area

$\rho, \rho C_p, \rho v_0 \neq f(r)$ or $f(z)$

Also reactor wall is insulated (No heat loss)

$\therefore T \neq f(r)$

$T = f(z)$

(i) Find the temp distribution in the z direction

Consider S_c in the rate of heat generation / volume due to chemical reaction:

$S_c = S_{c1} F(\theta)$ where $\theta = \frac{T - T_0}{T_1 - T_0}$

$T \to$ local temp

$T_0 \to$ initial condi

$S_{c1} \to$ constant of initial condition

Consider a strip of Δz thickness.

Balance eqn

$$\pi R^2 e_z \Big|_z - \pi R^2 e_z \Big|_{z+\Delta z} + \pi R^2 \cdot \Delta z \cdot S_c = 0$$

$$\frac{de_z}{dz} = S_c$$

z component of the $[\tau \cdot v]$ term *

$$e_z = \left(\tfrac{1}{2} \rho v_z^2 + \rho H\right) \cdot v_z + \{\tau_{zz} \cdot v_z\} + q$$

$$= \tfrac{1}{2} \rho v_z^2 v_z + \rho \hat{C_p} (T - T_0) \cdot v_z + \mu \frac{dv_z}{dz} \cdot v_z$$

$$+ \rho (P - P^0) v_z - k_{eff,zz} \frac{dT}{dz} = 0 \quad \text{expn}$$

* it'd be noted that $v_x = v_y = 0$ $\therefore \tau_{zx} \cdot v_x + \tau_{zy} \cdot v_y + [\tau_{zz} \cdot v_z] \to 0$

As
$V_z \neq f(z)$ also pressure grad. can be neglected

$\therefore \rho \hat{C_p} (T - T_0) v_z - k_{eff\,zz} \frac{dT}{dz} = S_c$ for zone II

$\rho \hat{C_p} v_z \frac{dT}{dz} = k_{eff\,zz} \frac{d^2T}{dz^2} + S_c$ for zone II

$z<0$ $\rho \hat{C_p} v_0 \frac{dT^I}{dz} = k_{eff\,zz} \frac{d^2T^I}{dz^2}$ — (I)

$0<z<L$ $\rho \hat{C_p} v_0 \frac{dT^{II}}{dz} = k_{eff\,zz} \frac{d^2T^{II}}{dz^2} + S_{c_1} f(\theta)$ — (II)

$z>L$ $\rho \hat{C_p} v_0 \frac{dT^{III}}{dz} = k_{eff\,zz} \frac{d^2T^{III}}{dz^2}$ — (III)

B.C.'s

(1) at $z=-\infty$ $T^I = T_1$

(2) at $z=0$ $T^{I} = T^{II}$

(3) $z=0$ $k_{eff} \frac{dT^I}{dz} = k_{eff} \frac{dT^{II}}{dz}$

(4) at $z=L$ $T^{II} = T^{III}$

(5) $z=L$ $k_{eff} \frac{dT^{II}}{dz} = k_{z} \frac{dT^{III}}{dz}$

(6) $z=\infty$ T^{III} = finite

For practical interest ~~damping~~ $k_{eff,zz}$ may be small compared to convection term ~~be~~

$\therefore k_{eff}=0$, (ensures plugflow)

for large $Pe' = Re.Pr$

consider $\xi = z/L$,

$N = \frac{S_{c_1} L}{\rho \hat{C_p} v_0 (T_1 - T_0)}$

$\rightarrow$ dimensionless heat generation

$\frac{D v \rho}{\mu} \cdot \frac{C_p \mu}{k}$

(104) Then from eqn (1) (2) & (3)

Zone I ($z < 0$) $\quad \frac{d\Theta^{I}}{dz} = 0 \Rightarrow \left\{ \rho \hat{C}_p v_0 \frac{dT^{I}}{dz} = 0 \right.$

$(T_1 - T_0) \dfrac{d\left(\frac{T^{I} - T_0}{T_1 - T_0}\right)}{L\, d(z/L)} = 0$

$\frac{d\Theta^{I}}{dz} = 0$

Zone II $\quad 0 < z < L \quad \frac{d\theta^{II}}{dz} = N f(\theta)$

Zone III

$z > L \quad \frac{d\theta^{III}}{dz} = 0$

We need three B.C to solve above eqns

I. $z = -\infty \quad \Theta^{I} = 1$

$z = 0 \quad \Theta^{I} = \Theta^{II}$

$z = 1 \quad \Theta^{II} = \Theta^{III}$

Soln

$\Theta^{I} = 1$ Zone-I

$\int_{\Theta^{I}}^{\Theta^{II}} \frac{1}{F(\Theta)} = N z$ Zone II

$\Theta^{III} = \Theta^{II} \Big|_{z=1}$ Zone III

$z = z/L \Rightarrow L/L = 1$

As an approximation

$F(\Theta) = \Theta^{m}$ { for small changes in temperatures if the reaction rate is insensitive to concentration }

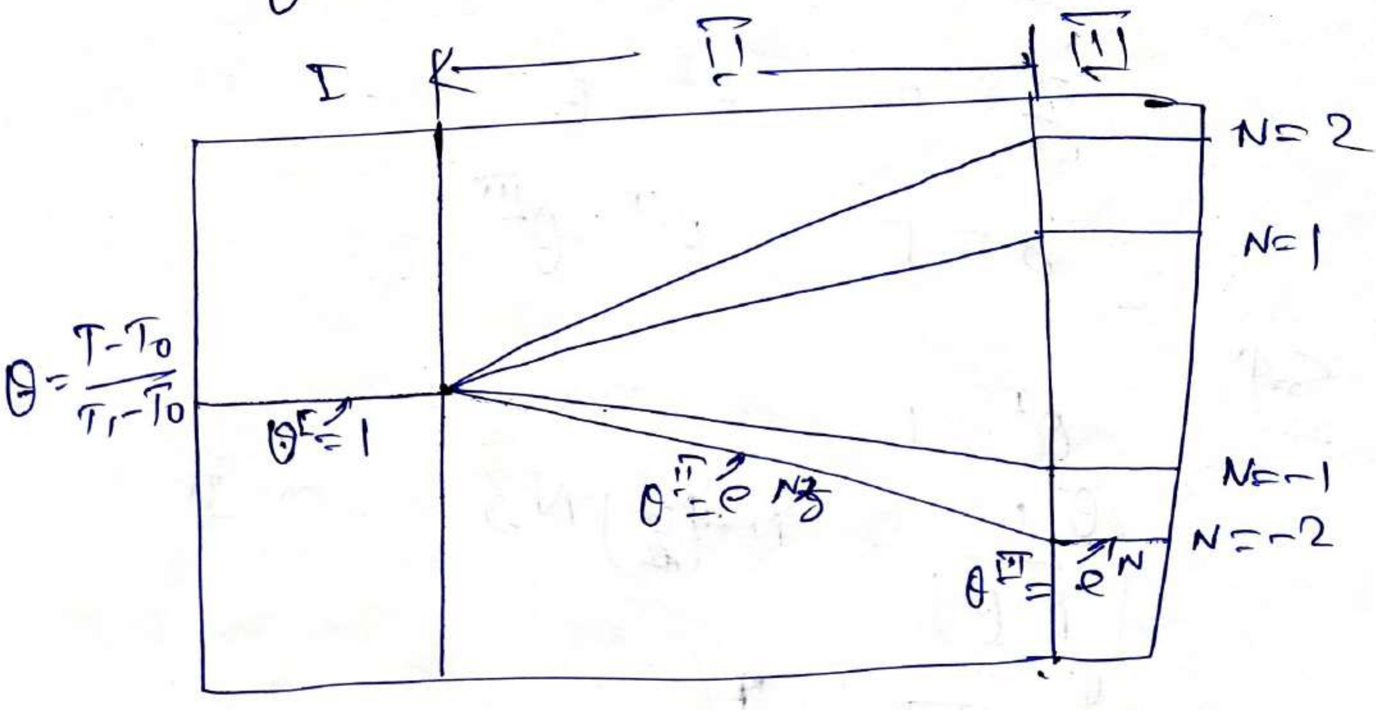
Thus we have

$\theta^{I} = 1$ Zone 1

$\theta^{II} = e^{N\zeta}$ Zone II

$\theta^{III} = \theta^{II}\big|_{\zeta=1} = e^{N\cdot(1)} = e^{N}$

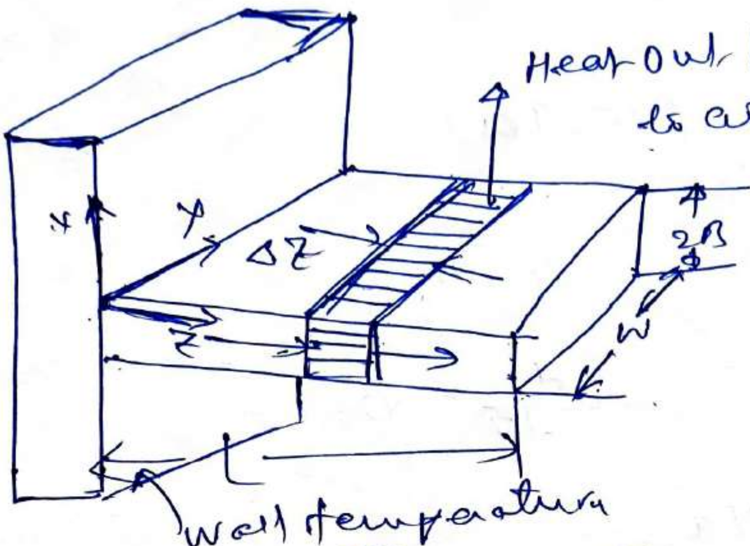
$\theta^{III} = e^{N}$ Zone III



$\zeta = z/L$

Heat conduction in a cooling fin

we'll find cooling fin efficiency



A single cooling fin with $B << L$, and $B << W$

$T_a \rightarrow$ ambient temp.

Actual situation

1. $T = f(x, y, z)$ — most important
2. heat is also lost from $2BW$
3. $h = f(\text{position})$

model

$T = f(z)$

No heat loss from the edges

$q_z = h(T - T_a)$ — constant & $T = f(z)$

Energy balance:

$$2BW\, q_z|_z - 2BW\, q_z|_{z+\Delta z} - h(2W\Delta z)(T - T_a) = 0$$

Division by $2BW\Delta z$ and taking the limit as Δz approaches zero gives

$$-\frac{dq_z}{dz} = \frac{h}{B}(T - T_a)$$

$\left(q_z = -\frac{k\,dT}{dz}\right)$ in which k is the thermal Conductivity of the metal.

$$\frac{d^2T}{dz^2} = \frac{h}{kB}(T - T_a)$$

B.C.1 at $z = 0$, $T = T_w$

B.C.2 at $z = L$ $\frac{dT}{dz} = 0$

Theta ← $\Theta = \frac{T - T_a}{T_w - T_a}$

(Zeta) ← $\xi = \frac{z}{L}$

$N^2 = \frac{hL^2}{kB}$ dimensionless HTC.

$\frac{d^2\Theta}{d\xi^2} = N^2\Theta$ with $\Theta|_{\xi=0} = 1$ and

$\frac{d\Theta}{d\xi}\Big|_{\xi=1} = 0$

The quantity N^2 may be $N^2 = \left(\frac{hL}{k}\right)\cdot\left(\frac{L}{B}\right)$
$= Bi\left(\frac{L}{B}\right)$

Solⁿ

$\Theta = \cosh N\xi - (\tanh N)\sinh N\xi$

$\Theta = \frac{\cosh N(1-\xi)}{\cosh N}$

$\eta = \frac{\text{actual rate of heat loss from the fin}}{\text{rate of heat loss from an isothermal fin at } T_w}$

(108)

~~$\eta =$ actual rate of heat loss~~

$$\eta = \frac{\int_0^W \int_0^L h\,(T - T_a)\,dz\,dy}{\int_0^W \int_0^L h\,(T_w - T_a)\,dz\,dy}$$

$$= \frac{\int_0^1 \theta \, d\xi}{\int_0^1 d\xi}$$

$$\eta = \frac{1}{\cosh N}\left(-\frac{1}{N}\sinh N\,(1-\xi)\right)\Big|_0^1$$

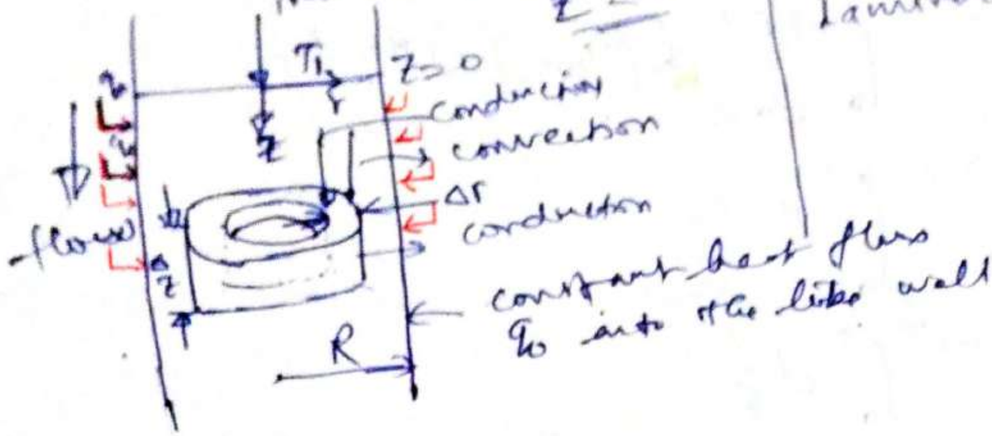
$$= \frac{\tanh N}{N}$$

in which N is dimensionless quantity

Forced Convection

Consider forced convection in a circular tube

Consider fluid has constant ($\mu, \rho, \hat{C}_p, k$), laminar flow



As the energy is being transported in the z and r direction consider a ring of fluid element of thickness Δr & length Δz.

(*) refer to shell 110

Energy balance:

Total energy in at $r = e_r|_r \cdot 2\pi r \cdot \Delta z$

out at $r+\Delta r = e_r|_{r+\Delta r} \cdot 2\pi(r+\Delta r)\Delta z$

$= 2\pi r \Delta z \cdot e_r|_{r+\Delta r}$

Total energy in at $z = e_z|_z \cdot 2\pi r \cdot \Delta r$

out at $z+\Delta z = e_z|_{z+\Delta z} \cdot 2\pi r \cdot \Delta r$

work done on fluid by gravity $= \rho g \cdot 2\pi r \Delta r \Delta z \cdot v_z$

Force $\cdot$ $\frac{m}{sec}$

= Energy/s.

In forced convection problem velocity profile is found first and then it is used to obtain the temperature profile

* Here consider the velocity profile is fully developed

$$\therefore v_z = \left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{4\mu L}\right) R^2 \left[1 - \left(\frac{r}{R}\right)^2\right]$$

$$= V_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

Energy balance

$$\frac{(r e_r)|_r - (r e_r)|_{r+\Delta r}}{\Delta r} + r \cdot \frac{e_z|_z - e_z|_{z+\Delta z}}{\Delta z} + \rho v_z g_z r = 0$$

or as $\Delta r \to 0$, $\Delta z \to 0$

$$-\frac{1}{r}\frac{\partial}{\partial r}(r \cdot e_r) - \frac{\partial e_z}{\partial z} + \rho v_z g_z = 0$$

$$e_r = q_r + \left(\tfrac{1}{2}\rho v^2 + \rho \hat{H}\right) v_r^{\,0} + \left(\tau_{rr} v_r^{\,0} + \tau_{r\theta} v_\theta^{\,0} + \tau_{rz} v_z\right)$$

$$v_z \tau_{rz} = -\mu\left(\frac{\partial v_z}{\partial r}\right) v_z$$

$$= -k\frac{\partial T}{\partial r} - \mu\frac{\partial v_z}{\partial r} \cdot v_z$$

$$e_z = q_z + \left(\frac{1}{2}\rho v^2 + \rho\hat{H}\right) v_z + \left(\tau_{zr} v_r^{\,0} + \tau_{z\theta} v_\theta^{\,0} + \tau_{zz} v_z^{\,0}\right)$$

$$= -k\frac{\partial T}{\partial z} + \left(\tfrac{1}{2}\rho v^2 + \rho\hat{H}\right) v_z$$

$\hat{H}$ can be determined from law of thermodynamics

$\hat{H} = f(T, P)$

$$d\hat{H} = \left(\frac{\partial \hat{H}}{\partial T}\right)_P dT + \left(\frac{\partial \hat{H}}{\partial P}\right)_T dP$$

$$= \hat{C}_p\, dT + \left[T\left(\frac{\partial S}{\partial P}\right)_T + V\left(\frac{\partial P}{\partial P}\right)_T\right] dP$$

Maxwell eq $d\hat{H} = T\,dS + V\,dP$

$$= \hat{C}_p\, dT + \left[T\left(-\frac{\partial \hat{V}}{\partial T}\right)_p + \hat{V}\right] dp$$

<u>For gas</u> consider Ideal gas law follows

$$PV = RT$$

$$\therefore \left(\frac{\partial V}{\partial T}\right) = \frac{R}{P} \Rightarrow \frac{RT}{P} = V$$

$$d\hat{H} = \hat{C}_p\, dT$$

<u>on integration</u>

$$\hat{H} - \hat{H}^0 = \hat{C}_p (T - T^0)$$

$\hat{C}_p$ → Assume $\neq f(T)$

<u>For ~~liquids~~ fluid of constant density</u>

First term will remain same

$$\hat{H} - \hat{H}^0 = \hat{C}_p (T - T^0) + \int_{p^0}^{p} \left[\hat{V} - T\left(\frac{\partial V}{\partial T}\right)_p\right] dp$$

Fluid is incompressible means ρ is constant

$$\rho = \frac{1}{\hat{V}} \quad \text{so } \hat{V} = \underline{\text{const}}$$

$$\hat{H} - \hat{H}^0 = \hat{C}_p (T - T^0) + \int_{p^0}^{p} \left[\hat{V} - T\left(\frac{\partial \hat{V}}{\partial T}\right)_p^{\nearrow 0}\right] dp$$

$$= \hat{C}_p (T - T^0) + \hat{V}(P - P^0)$$

$$= \hat{C}_p (T - T^0) + \frac{(P - P^0)}{\rho}$$

Let $\hat{H}^0 \Rightarrow 0$ for reference

$$\therefore \hat{H} = \hat{C}_p (T - T^0) + \frac{(P - P^0)}{\rho}$$

$$e_z = -k\frac{\partial T}{\partial z} + \left(\frac{1}{2}\rho v^2 + \rho \hat{C_p}(T-T^0) + (P-P^0)\right) v_z$$

Substituting the terms in the shell balance eqn.

$$-\frac{1}{r}\left(\frac{\partial}{\partial r}(r e_r)\right) - \frac{\partial e_z}{\partial z} + \rho v_z g_z = 0$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\left[-k\frac{\partial T}{\partial r} - \mu\frac{\partial v_z}{\partial r}\cdot v_z\right]\right)$$

$$-\frac{\partial}{\partial z}\left[-k\frac{\partial T}{\partial z} + \frac{1}{2}\rho v_z^2 v_z + \rho\hat{C_p}(T-T^0)v_z + (P-P^0)v_z\right] + \rho v_z g_z = 0$$

→ neglected (conduction in z direction is small)

→ 0: diff of this term will be zero as ρ const. & $v_z \neq f(z)\ \forall z$

Hence

$$\frac{1}{r}\frac{\partial}{\partial r}\left[kr\frac{\partial T}{\partial r} + \mu v_z \cdot r\frac{\partial v_z}{\partial r}\right] - \rho\hat{C_p}\frac{\partial T}{\partial z}\cdot v_z - v_z\frac{dP}{dz} + \rho v_z g_z = 0$$

$$\frac{1}{r}\left[kr\frac{\partial^2 T}{\partial r^2} + \mu v_z\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \mu r\frac{\partial v_z}{\partial r}\cdot\frac{\partial v_z}{\partial r}\right] - \rho\hat{C_p}v_z\frac{\partial T}{\partial z} - v_z\frac{\partial P}{\partial z} + \rho v_z g_z = 0$$

$$k\frac{\partial^2 T}{\partial r^2} + \frac{k}{r}\frac{\partial T}{\partial r} + \mu\frac{v_z}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \mu\left(\frac{\partial v_z}{\partial r}\right)^2 - v_z\rho\hat{C_p}\frac{\partial T}{\partial z} - v_z\frac{\partial P}{\partial z} + \rho v_z g_z = 0$$

→ 0 viscous heat negligible

$$k\frac{\partial^2 T}{\partial r^2} + \frac{k}{r}\frac{\partial T}{\partial r} - v_z\rho\hat{C_p}\frac{\partial T}{\partial z} + \left(\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) - \frac{\partial P}{\partial z} + \rho g_z\right)v_z = 0$$

→ 0: z-comp of velocity in N-S eqn and will be = 0 for this case check N-S eqn.

$$\boxed{\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = v_z\rho\hat{C_p}\frac{\partial T}{\partial z}} \qquad \text{——— (1)}$$

Desired expression.

Now v_z is a fn. of r as

$$v_z = v_{z\,max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

Thus

$$\rho \hat{C_p} V_{z\,max}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{\partial T}{\partial z} = \frac{k}{r}\left[\frac{\partial}{\partial r}\left(r\cdot\frac{\partial T}{\partial r}\right)\right] \quad -(2)$$

To solve this eqn an alternate method is given in Arihant heat transfer PP-259

[This same eqn can be reached using eqn of change for non-isothermal system]

Boundary Conditions

(1) at $r=0$; $\frac{dT}{dr} = 0$ Symmetry $\forall z$

or $T=$ finite

(2) at $r=R$ $\quad -k\frac{\partial T}{\partial r} = q_0$ uniform heat flux at wall ∴ continuity of heat flux.

(3) at $z=0$ $\quad T=T_1 \quad \forall r$

Soln of above eqn involves the use of dimensionless parameters

$$\Theta = \frac{T-T_1}{\frac{q_0 R}{k}} \;;\quad \xi = \frac{r}{R} \;;\quad \zeta = \frac{z}{\rho \hat{C_p} v_{z,max}\frac{R^2}{k}}$$

(Theta) (xi) (Zeta)

$\zeta = \frac{z}{R}\cdot\frac{\mu}{D V_{z\,max}\,\rho}\cdot\frac{k}{\mu C_p}$

$z\left[\zeta = \frac{z}{R}\cdot\frac{1}{Re}\cdot\frac{1}{Pr}\right]$

Re & Pr are important in Forced convection correlations

The choice for dimension less temp is from (2) & (3) B.C.

The eqn then Becomes

$$\left(1-\xi^2\right)\frac{\partial\Theta}{\partial\zeta} = \frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Theta}{\partial\xi}\right) \text{ with B.C.} \quad -(3)$$

at $\xi=0$; $\Theta=$ finite; at $\xi=1$, $\frac{\partial\Theta}{\partial\xi}=1$; at $\zeta=0$, $\Theta=0$

An asymptotic solution for the above eqn could be obtained for large ζ. As for large ζ the temperature profile as a function of ξ will not undergo further change with increasing ζ. Thus for large ζ

$$\Theta(\xi,\zeta) = C_0\zeta + \psi(\xi) \quad -(4)$$

where C_0 is a constant to be determined.

However this eqn (4) does not satisfy B.C. 3, but satisfies B.C. 1 and B.C. 2. Hence B.C. 3 needs to be changed.

{Note: if you put $\zeta=0$ in eqn 4 $\Theta\neq 0$}

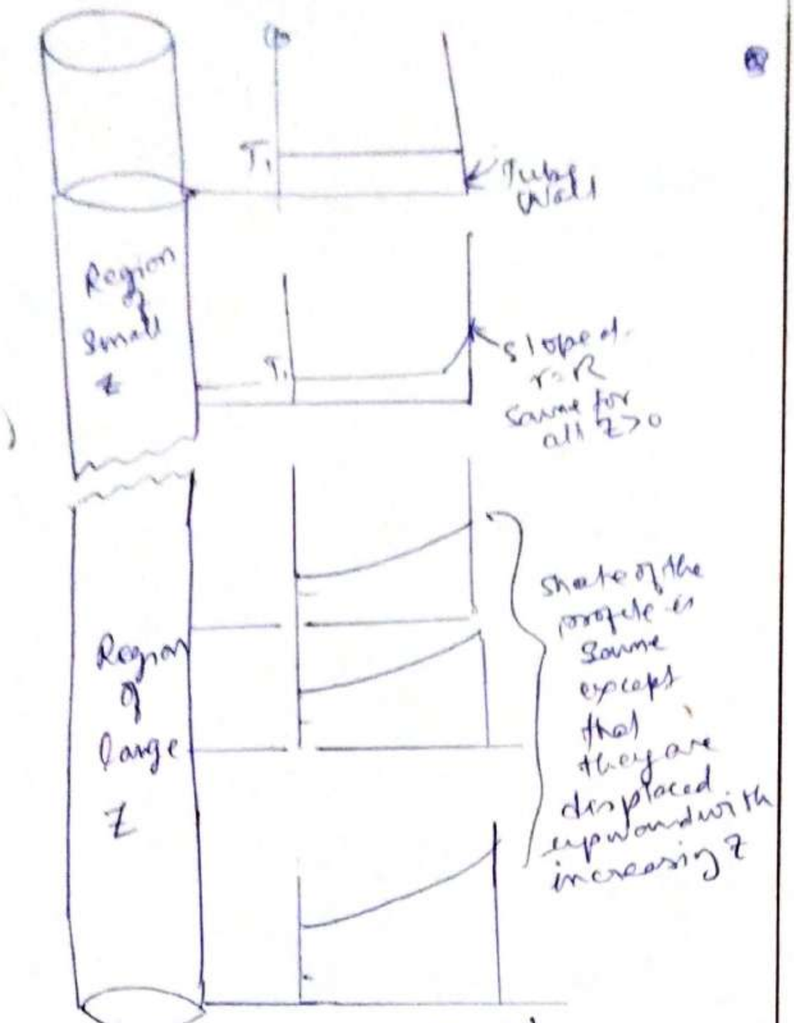


Figure: how the temperature would change when the tube wall is heating using a coil wrapped around the tube uniformelly

B.C.4

$$2\pi R z q_0 = \int_0^{2\pi}\int_0^R \rho \hat{C}_p (T - T_1) v_z r\,dr\,d\theta$$

or in dimensionless form.

$$\zeta = \int_0^1 \Theta(\xi, \zeta)(1-\xi^2)\xi\, d\xi \quad — (5)$$

i.e. Energy supplied over a distance ζ is = the (energy leaving at ζ - energy entering at $\zeta = 0$)

Substituting eqn (4) into eqn (3)

$$\frac{1}{\xi}\frac{d}{d\xi}\left(\xi \frac{d\psi}{d\xi}\right) = C_0(1-\xi^2)$$

which gives on twice integration.

$$\Theta(\xi, \zeta) = C_0\zeta + C_0\left(\frac{\xi^2}{4} - \frac{\xi^4}{16}\right) + C_1 \ln \xi + C_2$$

Using B.C.s (1), (2) and (4)

the constants are $C_1 = 0$ from B.C. 1

$C_0 = 4$ — " — B.C. 2

$C_2 = -7/24$ from condition 4

Thus

$$\Theta = 4\zeta + \xi^2 - \frac{1}{4}\xi^4 - \frac{7}{24}$$

Valid for large ζ: $\zeta \to \infty$

Arithmatic avg temp.

$$\langle T \rangle = \frac{\int_0^{2\pi}\int_0^R T(r,z)\, r\,dr\,d\theta}{\int_0^{2\pi}\int_0^R r\,dr\,d\theta} = T_1 + \left(4\zeta + \frac{7}{24}\right)\frac{q_0 R}{k}$$

Bulk avg. temp or mixing cup temp.

$$T_b = \frac{\langle v_z T\rangle}{\langle v_z\rangle} = \frac{\int_0^{2\pi}\int_0^R v_z(r)\, T(r,z)\, r\,dr\,d\theta}{\int_0^{2\pi}\int_0^R v_z(r)\, r\,dr\,d\theta}$$

$$T_b = T_1 + 45\frac{q_0 R}{K}$$

Local Heat Transfer Driving Force, ~~Tw-Tb~~ $T_W - T_b$

~~Tw-Tb~~

@ $r = R$, $T = T_W$

$$\therefore \frac{T - T_1}{\frac{q_0 R}{K}} = 45 + \left(\frac{r}{R}\right)^2 - \frac{1}{4}\left(\frac{r}{R}\right)^4 - \frac{7}{24}$$

$$T_W - T_1 = 45\frac{q_0 R}{K} + \frac{q_0 R}{K}\left[1 - \frac{1}{4} - \frac{7}{24}\right]$$

($T_1 + 45\frac{q_0 R}{K}$ ↳ T_b)

$$T_W - T_b = \frac{q_0 R}{K}\left[\frac{11}{24}\right] = f(r) \text{ only.}$$

or

$$\frac{q_0}{K(T_W - T_b)} \cdot R = \frac{24}{11} \Rightarrow \frac{q_0}{K(T_W - T_b)} \cdot D = \frac{48}{11}$$

$$\because q_0 = h(T_W - T_b)$$

$$\boxed{\frac{hD}{K} = \frac{48}{11}} \Rightarrow \text{limiting value of Nusselt Number}$$

The Nusselt number depends upon Re & Pr in case of Forced convection.

Refer to Page No. 235-247
Geankoplis.

Heat Transfer – (i) Fourier's Law
(ii) Notes on Th. Conductivity
(iii) Derivations - Parallel wall, Cylindrical wall, Composite wall
and Numericals based on them.

The equation of change for Nonisothermal system.

Law of Conservation of energy which is an extension of first law of thermodynamics will be applied over a differential volume to obtain the energy equation.

First law of thermodynamics

$\Delta U = Q + W$

Q involves : entering and leaving K.E & I.E. due to conv. & cond.

W → { work done by Body forces like, gravity
W → { Surface forces such as, pressure, viscous force.

Thus the General expression for the energy conservation thus becomes

Rate of increase of K.E. & I.E. = Net rate of K.E. & I.E. addition by conv. transport + Net rate of heat addition by molecular transport conduction + rate of work done on system by molecular mechanism i.e. by stresses (P, τ etc.) + rate of work done due to external forces e.g. by gravity — (1)

Mathematically

$$L.H.S. = \Delta x \cdot \Delta y \cdot \Delta z \cdot \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho \hat{U}\right)$$

$\rho\hat{U}$ → I.E./vol.

$\hat{U}$ → Energy / mass

$\frac{1}{2}\rho v^2 = \frac{1}{2}\rho(v_x^2 + v_y^2 + v_z^2)$ → K.E/vol.

The first three terms ~~of the eqn~~ R.H.S of the eqn are present in $\underline{e}$ i.e. combined energy flux vector.

Energy entering the volume element $\Delta x \Delta y \Delta z$

(117)

$$= \Delta y \Delta z (e_x|_x - e_x|_{x+\Delta x}) + \Delta z \Delta x (e_y|_y - e_y|_{y+\Delta y}) + \Delta x \Delta y (e_z|_z - e_z|_{z+\Delta z}) \quad \text{——} (2)$$

work done on fluid due to gravity force (external forces)

$$= \rho \Delta x \Delta y \Delta z (g \cdot v) = \rho \Delta x \Delta y \Delta z (g_x v_x + g_y v_y + g_z v_z) \quad \text{——} (3)$$

From (1), (2) & (3)

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho \hat{U}\right) = -\left(\frac{\partial e_x}{\partial x} + \frac{\partial e_y}{\partial y} + \frac{\partial e_z}{\partial z}\right) + \rho(g_x v_x + g_y v_y + g_z v_z)$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho \hat{U}\right) = -(\nabla \cdot e) + \rho(v \cdot g)$$

Extending vectore

$$e = \left(\frac{1}{2}\rho v^2 + \rho \hat{U}\right)v + q + pv + [\tau \cdot v]$$

Thus

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho \hat{U}\right) = \left(\nabla \cdot \left(\frac{1}{2}\rho v^2 + \rho \hat{U}\right)v\right) - (\nabla \cdot q) - (\nabla \cdot pv) - (\nabla \cdot [\tau \cdot v]) + \rho(v \cdot g)$$

Terms		Description
L.H.S.		rate of Increase of K.E. & I.E. per unit vol.
R.H.S. (1)	—	rate of energy addition due to conv. transport per vol.
(2)	—	conductive transport (Molecular transport)
(3)	—	work done by pressure force / vol.
(4)	—	work done by viscous force. τ has same unit as p
(5)	—	work done by gravity force.

$p \rightarrow$ Force / Area
$v \rightarrow m/s$
$\nabla \rightarrow 1/m$

$\frac{\text{Force} \times \frac{m}{s}}{\text{Area}} \times \frac{1}{m}$ — $\frac{\text{Energy rate}}{\text{vol.}}$

Above eqn doesn't include nuclear, radioactive, electromagnetic or chemical forms of energy.

Special Forms of Energy eqn:

The energy eqn is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho \hat{U}\right) = -\left(\nabla\cdot\left(\frac{1}{2}\rho v^2 + \rho\hat{U}\right)\bar{v}\right) - (\nabla\cdot\bar{q}) - (\nabla\cdot p\vec{v}) - (\nabla\cdot[\bar{\tau}\cdot\bar{v}]) + \rho(\bar{v}\cdot g)$$

From this we subtract the Mechanical energy eqn

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) = -\left(\nabla\cdot\frac{1}{2}\rho v^2\vec{v}\right) - (\nabla\cdot p\vec{v}) - p(-\bar{\nabla}\cdot\vec{v}) - (\nabla\cdot[\tau\cdot\bar{v}]) - (-\bar{\tau}:\bar{\nabla}\bar{v}) + \rho(\bar{v}\cdot\bar{g})$$

$$\frac{\partial}{\partial t}(\rho\hat{U}) = -(\bar{\nabla}\cdot\rho\hat{U})\bar{v} - (\nabla\cdot\bar{q}) + p(-\bar{\nabla}\cdot\bar{v}) + (-\bar{\tau}:\nabla\bar{v})$$

irreversible rate of I.E. increase / vol by viscous Dissipation

or ~~For incompressible fluid~~

$$\rho\frac{D\hat{U}}{Dt} = -(\bar{\nabla}\cdot\bar{q}) - p(\bar{\nabla}\cdot\vec{v}) - (\bar{\tau}:\bar{\nabla}\bar{v})$$

→ More explanation in Appendix A.4

Reversible rate of internal energy increase per unit volume by compression.

$\rightarrow \tau:\nabla v = \tau_{xx}\frac{\partial v_x}{\partial x} + \tau_{xy}\frac{\partial v_x}{\partial y} + \tau_{xz}\frac{\partial v_x}{\partial z} + \tau_{yx}\frac{\partial v_y}{\partial x} + \tau_{yy}\frac{\partial v_y}{\partial y} + \tau_{yz}\frac{\partial v_y}{\partial z} + \tau_{zx}\frac{\partial v_z}{\partial x} + \tau_{zy}\frac{\partial v_z}{\partial y} + \tau_{zz}\frac{\partial v_z}{\partial z}$

Further

$$\hat{U} = \hat{H} - pV = \hat{H} - (P/\rho)$$

$\therefore$

$$\frac{D\hat{U}}{Dt} = \frac{D\hat{H}}{Dt} - \frac{1}{\rho}\frac{DP}{Dt}$$

$\therefore$

$$\rho\frac{D\hat{H}}{Dt} = -(\bar{\nabla}\cdot\bar{q}) - (\bar{\tau}:\bar{\nabla}\bar{v}) + \frac{DP}{Dt} \quad \text{— (1)}$$

$$\rho\frac{D\hat{H}}{Dt} = \rho\hat{C}_p\frac{DT}{Dt} + \rho\left[\hat{V} - T\left(\frac{\partial\hat{V}}{\partial T}\right)_P\right]\frac{DP}{Dt}$$

} from eqn 9.8-7

$$= \rho\hat{C}_p\frac{DT}{Dt} + \rho\left[\frac{1}{\rho} - T\left(\frac{\partial 1/\rho}{\partial T}\right)_P\right]\frac{DP}{Dt}$$

$$\rho \frac{D\hat{H}}{Dt} = \rho \hat{C}_p \frac{DT}{Dt} + \left[1 + \left(\frac{\partial \ln \rho}{\partial \ln T}\right)_p\right] \frac{DP}{Dt} \quad \text{— (2)}$$

Substituting this value into eqn (1) we have

$$\boxed{\rho \hat{C}_p \frac{DT}{Dt} = -(\bar{\nabla} \cdot \dot{q}) - (\bar{\tau} : \bar{\nabla} \dot{v}) - \left(\frac{\partial \ln \rho}{\partial \ln T}\right)_p \frac{DP}{Dt}}$$

— eqn of chang for temperature

Refer to Appendix B.8 Bird

When fourieis law is used $-(\nabla \cdot q) = (\nabla \cdot K \nabla T)$

if K is constant then $= (K \cdot \nabla^2 T)$

viscous heating term ($\tau : \nabla v$) is neglected since it is important only when the velo grad. is large.

Special cases :

(i) For ideal gas $\left(\frac{\partial \ln \rho}{\partial \ln T}\right)_p = -1$ So,

$$\rho \hat{C}_p \frac{DT}{Dt} = K \nabla^2 T + \frac{DP}{Dt}$$

(ii) For fluid flowing in a constant pressure system

$\frac{DP}{Dt} = 0$ ∴

$$\boxed{\rho \hat{C}_p \frac{DT}{Dt} = K \nabla^2 T}$$

(iii) For fluid with constant density

$$\left(\partial \ln \rho / \partial \ln T\right)_p = 0$$

$$\therefore \boxed{\rho \hat{C}_p \frac{DT}{Dt} = k \nabla^2 T}$$

(iv) For a stationary solid, v is zero hence

$$\boxed{\rho \hat{C}_p \frac{\partial T}{\partial t} = K \nabla^2 T} \rightarrow \text{Fourier's eqn}$$

Relevant dimensionless groups:

$Re = \left[\frac{l_0 v_0 \rho}{\mu}\right]$ = Reynolds number

$Pr = \left[\frac{c_p \mu}{k}\right] = \frac{\nu}{\alpha}$ = Prandtl number

$Gr = \left[g\beta (T_1 - T_0) l_0^3 / \nu^2\right]$ = Grashoff Number

$Br = \left[\mu v_0^2 / k (T_1 - T_0)\right]$ = Brinkman Number

$Pe = Re\,Pr$ = Peclet number

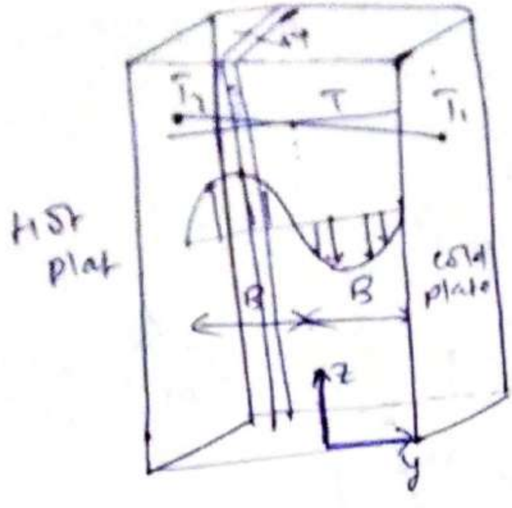
$Ra = Gr\,Pr$ = Rayleigh Number

$Ec = Br / Pr$ = Eckert Number

Refer to Table 11.5-3

A

free convection Problem:



Ex. flow pattern b/w two parallel plates maintained at different temperatures

fluid of density ρ and viscosity μ is located b/w the plates

It is assumed that temp difference is sufficiently small.

* System is closed at the top & bottom.

Due to the temp diff. the fluid at hot end rises and that on cold end descends and the velocity profile as shown develops

* The plate are assumed to be very tall so that end effects can be neglected.
* Temperature is a fn. of 'y' alone.

Select a shell a thickness of Δy to make energy balance.

in 'y' direction there is no convection and heat transfer is only by conduction (neglect the viscous heating term)

$$\therefore \quad -\frac{dq_y}{dy} = 0 \quad \text{or} \quad k\frac{d^2T}{dy^2} = 0$$

at $y = -B, \quad T = T_2, \quad \therefore \quad y = +B \quad T = T_1$

$$\therefore \quad \boxed{T = \bar{T} - \tfrac{1}{2}\Delta T \, y/B} \qquad \bar{T} = \tfrac{1}{2}(T_1 + T_2)$$

$$\Delta T = T_2 - T_1$$

Now let's find velocity distribution

Momentum shell balance over the Δy slab

$\phi_{xz}, \phi_{yz}, \phi_{zz}$

$v_x = v_y = 0$

$$\phi_{xz} = \rho v_x v_z + p\cdot 0 + \left(-\mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)\right) = 0$$

$$\phi_{yz} = \rho v_y v_z + p\cdot 0 + \left[-\mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right)\right] = -$$

$$\phi_{zz} = \rho v_z^2 + p + \left[-2\mu \frac{\partial v_z}{\partial z}\right]$$

on making balance

$$\mu \frac{d^2 v_z}{dy^2} = \frac{dp}{dz} + \rho g$$

$\mu \rightarrow$ assumed constant

$\rho = f(T)$ $\because$ Natural convection

As the ΔT is small change in ρ will be small hence ρ can be expanded about $\bar{T}$ using Taylor series

$$\therefore \rho = \rho\big|_{T=\bar{T}} + \frac{d\rho}{dT}\bigg|_{T=\bar{T}} (T-\bar{T}) + \cdots$$

Boussinesq approximation $\rightarrow$

$$\rho = \bar{\rho} - \bar{\rho}\bar{\beta}(T-\bar{T})$$

$\beta \rightarrow$ volume expansion coefficient

$$\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = \frac{1}{(1/\rho)}\left(\frac{\partial (1/\rho)}{\partial T}\right)_p = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$

$\therefore$

$$\mu \frac{d^2 v_z}{dy^2} = \frac{dp}{dz} + \left[\bar{\rho} - \bar{\rho}\bar{\beta}(T-\bar{T})\right] g$$

~~$= \frac{dp}{dz} + \left[\bar{\rho} - \bar{\rho}\bar{\beta}\, \tfrac{1}{2}\Delta T (y/B)\right] g$~~

Note that the Temperature change is small hence the density change will be small

Assume that at $\bar{T} = \frac{1}{2}(T_2 + T_1)$

$$\rho = \bar{\rho}$$

Using Taylor series expansion ρ can be then expanded around $\bar{T}$ as

$$\rho = \bar{\rho}\Big|_{T=\bar{T}} + \frac{d\rho}{dT}\Big|_{T=\bar{T}} (T - \bar{T})$$

$$= \bar{\rho} - \bar{\rho}\bar{\beta}(T - \bar{T})$$

$(\rho, T) \quad (\bar{\rho}, \bar{T})$

$\bar{\rho}$, & $\bar{\beta}$ are the density and the volume expansion coefficient at $\bar{T}$

β is defined as

$$\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p = \frac{1}{(1/\rho)}\left(\frac{\partial (1/\rho)}{\partial T}\right)_p$$

$$= -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$

Therefore

$$\mu \frac{d^2 v_z}{dy^2} = \frac{dp}{dz} + \left(\bar{\rho} - \bar{\rho}\bar{\beta}(T - \bar{T})\right) g$$

$$\mu \frac{d^2 v_z}{dy^2} = \left(\frac{dp}{dz} + \rho g\right) - \bar{\rho} g \bar{\beta}(T - \bar{T})$$

viscous force — pressure force — gravity force — Buoyancy force

But $T = \bar{T} - \frac{1}{2}\Delta T \frac{y}{B}$ ∴

$$\mu \frac{d^2 v_z}{dy^2} = \left(\frac{dP}{dz} + \bar{\rho} g\right) - \bar{\rho} g \bar{\beta} \left(\bar{T} - \tfrac{1}{2}\Delta T \tfrac{y}{B} - \bar{T}\right)$$

$$= \left(\frac{dP}{dz} + \bar{\rho} g\right) + \tfrac{1}{2} \bar{\rho} g \bar{\beta} \Delta T \, (y/B)$$

B.C.'s

(1) at $y = -B$, $v_z = 0$,

(2) at $y = +B$, $v_z = 0$

~~$v_z = \frac{(\bar{\rho} g \bar{\beta} \Delta T) B^2}{12\mu} \left[(y/B)^3 - (y/B) \right] + B^2$~~

$$\mu \frac{dv_z}{dy} = \left(\frac{dP}{dz} + \bar{\rho} g\right) \cdot y + \tfrac{1}{4} \bar{\rho} g \bar{\beta} \left(\Delta T \cdot \tfrac{y^2}{B}\right) + C_1$$

$$\mu v_z = \left(\frac{dP}{dz} + \bar{\rho} g\right) \cdot \frac{y^2}{2} + \frac{1}{12} \bar{\rho} g \bar{\beta} \left(\Delta T \frac{y^3}{B}\right) + C_1 y + C_2$$

from (1) B.C.

$$0 = \left(\frac{dP}{dz} + \bar{\rho} g\right) \cdot \frac{B^2}{2} - \frac{1}{12} \bar{\rho} g \bar{\beta} \cdot \Delta T \cdot B^2 - C_1 B + C_2 \quad \text{— (X)}$$

II B.C

$$0 = \left(\frac{dP}{dz} + \bar{\rho} g\right) \cdot \frac{B^2}{2} + \frac{1}{12} \bar{\rho} g \bar{\beta} \left(\Delta T B^2\right) + C_1 B + C_2 \quad \text{— (Y)}$$

(X + Y)

$$C_2 = -\left(\frac{dP}{dz} + \bar{\rho} g\right) \cdot \frac{B^2}{2}$$

$\therefore$ from X

$$C_1 = -\frac{1}{12} \bar{\rho} g \bar{\beta} \Delta T \cdot B$$

~~$\frac{dp}{dz}+\bar{\rho}g$~~ $c_1 = -\frac{1}{12}\bar{\rho} g \bar{\beta} \Delta T \underline{B}$ (12)

$\therefore v_z = \frac{1}{2\mu}\left(\frac{dp}{dz}+\bar{\rho}g\right)(y^2-B^2)$

$+ \frac{1}{12\mu}\bar{\rho} g \bar{\beta}\Delta T \frac{y^3}{B} - \frac{1}{12}\bar{\rho} g\bar{\beta}\Delta T B.y$

$v_z = \frac{B^2}{2\mu}\left(\frac{dp}{dz}+\bar{\rho}g\right)\left(\left(\frac{y}{B}\right)^2-1\right)$

$+ \frac{1}{12\mu}\bar{\rho} g\bar{\beta}\Delta T\, y.B\left(\frac{y^2}{B^2}-1\right)$

$\therefore v_z = \frac{1}{12\mu}\bar{\rho} g \bar{\beta}\Delta T B^2\left[\left(\frac{y}{B}\right)^3-\left(\frac{y}{B}\right)\right] + \frac{B^2}{2\mu}\left(\frac{dp}{dz}+\bar{\rho}g\right)\left[\left(\frac{y}{B}\right)^2-1\right]$

<u>Mass Balance</u>

The net mass flow in the z direction is zero

$\int_{-B}^{+B}\rho v_z\,dy = 0$ $\qquad \frac{dp}{dz} = -\bar{\rho}g$

substitute

$\rho = \bar{\rho} - \bar{\rho}\bar{\beta}\left(\frac{1}{2}\Delta T\, y/B\right)$

& v_z from above eqn

Just remember it is $(+)B$ & $-B$ in the ~~even~~ ~~terms~~ terms with limits so the ~~square~~ integration will even power of y after integration will cancel out and only odd power term in second term will remain which yields v_z expression. And that yields

$\left(\frac{dp}{dz}+\bar{\rho}g\right)$. coefficient = 0

$\therefore \boxed{\frac{dp}{dz}+\bar{\rho}g = 0}$

Therefor the expression for v_3 becomes.

$$v_3 = \frac{(\bar{\rho} g \bar{\beta} \Delta T) B^2}{12\mu}\left[\left(\frac{y}{B}\right)^3 - \left(\frac{y}{B}\right)\right]$$

avg velocity of upward moving stream

$$\langle v_3 \rangle = \frac{\int_{-B}^{0} v_3\, dy \cdot W}{(-B \cdot W)} = \frac{\frac{W \bar{\rho} g \bar{\beta} \Delta T B^2}{12\mu}\left[\frac{y^4}{4B^3} - \frac{y^2}{2B}\right]_{-B}^{0}}{(-B \cdot W)}$$

$$= \frac{\frac{\bar{\rho} g \bar{\beta} \Delta T B^2}{48\mu}\left[\frac{1}{B} B\right] \cdot W}{-B \cdot W} = \frac{1}{48} \frac{\bar{\rho} g \bar{\beta} \Delta T B^2}{\mu}$$

These expression for v_3 shows that fluid motion is a consequence of buoyant force associated with the temperature gradient.

Let's define a dimensionless velocity

$$V_3 = \frac{B v_3 \bar{\rho}}{\mu} \quad \& \quad Y = (y/B)$$

thus

$$V_3 = \frac{1}{2} Gr\,(Y^3 - Y)$$

where Grashoff number = Gr

$$= \left[\frac{(\bar{\rho}^2 g \bar{\beta} \Delta T) B^3}{\mu^2}\right]$$

~~$\Delta \rho = \rho_1 - \rho_2$~~

$$Gr = \frac{\bar{\rho} g B^3 \bar{\rho} \bar{\beta}}{\mu^2}(T_2 - T_1) = \frac{\bar{\rho} g B^3}{\mu^2}\left(\bar{\rho}\bar{\beta}\left[(T_2 - \bar{T}) - (T_1 - \bar{T})\right]\right)$$

$$= \frac{\bar{\rho} g B^3}{\mu^2}\left[\bar{\rho}\bar{\beta}\Delta T_2 - \bar{\rho}\bar{\beta}\Delta T_1\right]$$

$$= \frac{\bar{\rho} g B^3}{\mu^2}\left[\underbrace{\bar{\rho} - \bar{\rho}\bar{\beta}\Delta T_1}_{\rho_1} - \underbrace{(\bar{\rho} - \bar{\rho}\bar{\beta}\Delta T_2)}_{\rho_2}\right]$$

$$Gr = \frac{\bar{\rho} g B^3}{\mu^2}\Delta\rho$$

$\Delta\rho = \rho_1 - \rho_2$
Note
$\Delta T = T_2 - T_1$

Assignment: Refer to example problem 11.5-1 and 11.5-2 of Bird.

~~UNSTEADY STATE HEAT TRANSFER in solids~~

For solids the governing heat transfer eq^n.

$$\rho \hat{C}_p \frac{\partial T}{\partial t} = \nabla \cdot q$$

$q = k \nabla T$

$$= \nabla \cdot k \nabla T$$

$k \rightarrow$ isotropic

assumed $k \neq f(T)$

$$= k \nabla^2 T$$

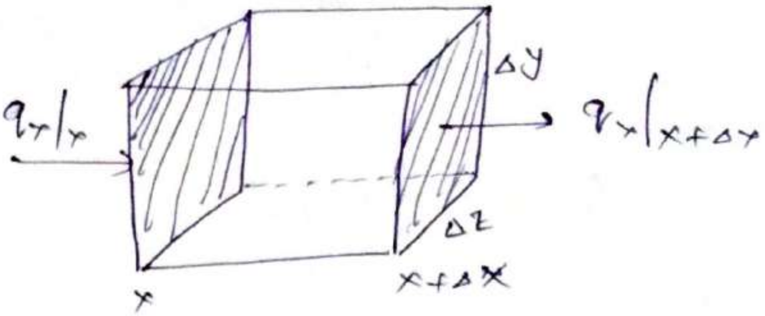
$$\frac{\partial T}{\partial t} = \underbrace{\frac{k}{\rho \hat{C}_p}}_{\alpha} \nabla^2 T \Rightarrow \frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

~~Heating a Semi-infinite slab~~

Unsteady-state heat transfer

Basic eqn

consider a cube of dimension $\Delta x, \Delta y, \& \Delta z$,



$$q_x = -kA\frac{\partial T}{\partial x}$$ for heat conduction in x direct.

Heat Balance

rate of heat (input − output) + generation = Acc.

$$\left(q_x|_x - q_x|_{x+\Delta x}\right) + \dot{q}\,(\Delta x.\Delta y.\Delta z) = \Delta x.\Delta y.\Delta z.\rho\, C_p \frac{\partial T}{\partial t}$$

$\dot{q} \rightarrow$ rate of Heat generation/vol.

~~$$\therefore\ \dot{q} - \frac{\partial q_x}{\partial x} = \rho C_p \frac{\partial T}{\partial t}$$~~

~~if $\dot{q} = 0$ then~~

~~$\rho C_p \frac{\partial T}{\partial t} = -\frac{\partial q_x}{\partial x}$~~

~~$$\rho C_p \frac{\partial T}{\partial t} =$$~~

$$\left(k\frac{\partial T}{\partial x}\Big|_x - k\frac{\partial T}{\partial x}\Big|_{x+\Delta x}\right).\Delta y.\Delta z + \Delta x\,\Delta y\,\Delta z\,\dot{q} = \Delta x\,\Delta y.\Delta z.\rho C_p\frac{\partial T}{\partial t}$$

$$\dot{q} + k\frac{\partial^2 T}{\partial x^2} = \rho c_p \frac{\partial T}{\partial t}$$

or

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_p}$$

$$= \alpha \frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_p}$$

k, ρ, c_p are assumed constant.

<u>SI Units</u> $\alpha \to m^2/s$, $T \to K$, $t \to s$, $k = W/m.K$

$\rho \to kg/m^3$, $\dot{q} = W/m^3$, $c_p = J/kg.K$

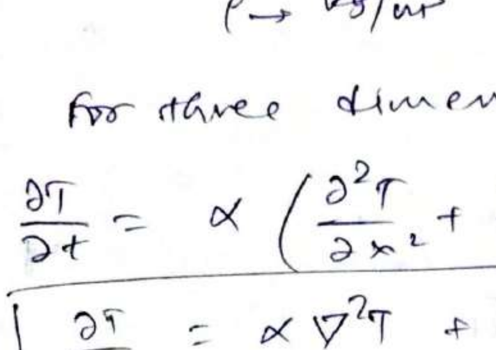
For three dimension case

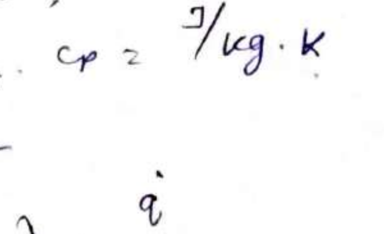
$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{\dot{q}}{\rho c_p}$$

$$\boxed{\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{\dot{q}}{\rho c_p}}$$

<u>Heating a semi-infinite slab</u>

A solid material occupying the space from $y=0$, to $y=\infty$ has initial temp T_0 and at $t=0$ temp at $y=0$ is raised to T_1 and maintained at it. Obtain the temp profile





at $t<0$ $T_{y=0} = T_0$ <u>I.C</u>

$t=0$ & $\forall >0$ $T_{y=0} = T_1$ <u>B.C</u>

Find <u>$T(y,t)$</u>

<u>Sol</u>^{n} define $\Theta = \frac{T - T_0}{T_1 - T_0}$

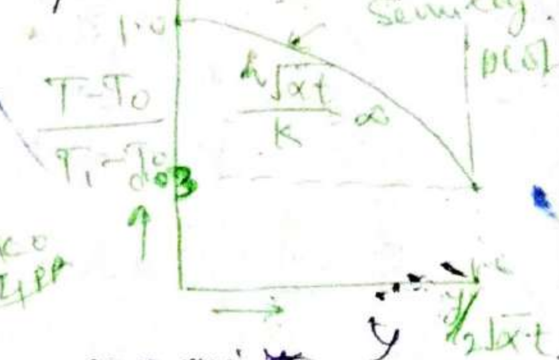
I.C. $t \le 0$ $\Theta = 0$ $\forall y$

B.C. 1 $y = 0$ $\Theta = 1$ $\forall t > 0$

$y = \infty$ $\Theta = 0$ $\forall t > 0$

$$\Theta = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{y}{\sqrt{4\alpha t}}} \exp(-\eta^2)\, d\eta$$

[look into previous momentum eqn]



$$\therefore \frac{T-T_0}{T_1-T_0} = 1 - \text{erf} \frac{y}{\sqrt{4\alpha t}}$$

plot $\frac{T-T_0}{T_1-T_0}$ ~~dim~~ on x axis $\frac{y}{\sqrt{4\alpha t}}$

when $\frac{y}{\sqrt{4\alpha t}} = 2$ then $\text{erf} \frac{y}{\sqrt{4\alpha t}} = 0.99$ $\boxed{\frac{T-T_0}{T_1-T_0} = 0.01}$

$\therefore y = 4\sqrt{\alpha t}$

$\boxed{\delta_T = 4\sqrt{\alpha t}}$ thermal penetration thickness.

That means for distances $y > \delta_T$ the temperature has change by less than 1% of $T_1 - T_0$

wall heat flux

$$q_y|_{y=0} = -k \frac{\partial T}{\partial y}\Big|_{y=0} = \frac{k}{\sqrt{\pi \alpha t}} (T_1 - T_0)$$

$$\boxed{q_y|_{y=0} \propto t^{-1/2}, \quad \& \quad \delta_T \propto t^{1/2}}$$

Read Pg no. 330 - 336 geankoplis